Real Analysis Portfolio

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Contents

1 The Field Axioms

4. Uniqueness of Additive Inverses

Let x be a real number. Prove that there is only one real number y such that $x + y = 0$.

For $x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{R}$, let $x_1 + y_1 = x_2 + y_2$.

$$
\begin{aligned}
x_1 + (-x_1) + y_1 &= x_2 + (-x_2) + y_2 && \text{Add. Inv. (2d)} \\
0 + y_1 &= 0 + y_2 && \text{Add. ID (2c)} \\
y_1 &= y_2
\end{aligned}
$$

$\therefore\ y_1 = y_2$ so the additive identity must be unique.

■

5. No. 4 (Cont.)

Prove that if a, b, and c are real numbers such that $a + b = 0$ and $a + c = 0$, then $b = c$.
Because $a + b = 0$, b must be the Additive ID of a according to 2c, which is unique, according to 4. Likewise, because $a + c = 0$, c too must be the Additive ID of a according to 2c, which is unique, according to 4. Since both b and c are the additive identities of a, and an additive identity must be unique, $b = c$.

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6. Multiplication by Zero

Prove that if x is a real number then $x \cdot 0 = 0$.

$$
\begin{aligned}
x \cdot 0 &= x \cdot 0 \\
&= x \cdot 0 + 0 && \text{Add. Id (2c)} \\
&= x \cdot 0 + (x - x) && \text{Add. Id (2c)} \\
&= (x \cdot 0 + x) - x && \text{Assoc. (2b)} \\
&= x(0 + 1) - x && \text{Distrib. (2e)} \\
&= x(1) - x && \text{Add. Id (2c)} \\
&= x - x && \text{Mult. Id (2c)} \\
&= 0 && \text{Add. Id (2c)}
\end{aligned}
$$

$\therefore\ x \in \mathbb{R}^+ \ \Rightarrow\ x \cdot 0 = 0$

■

7. Assoc. & Comm. with (-1)(x)(y)

Prove that if x and y are real numbers then $(-x)y = -(xy) = x(-y)$.

$(-x)y$	$= (-1 \cdot x)y$	Mult. by -1 (8)
	$= -1(x \cdot y)$	Assoc. (2b)
	$= -(xy)$	Mult. by -1 (8)
$(-x)y$	$= (-1 \cdot x)y$	Mult. by -1 (8)
	$= (x \cdot -1)y$	Comm. (2a)
	$= x(-1 \cdot y)$	Assoc. (2b)
	$= x(-y)$	Mult. by -1 (8)

$\therefore\ x, y \in \mathbb{R} \ \Rightarrow\ (-xy) = -(xy) = x(-y)$

■

8. Multiplication by -1

Prove that if x is any real number then (-1)x = -x.

$-1 \cdot x$	$= -1 \cdot x + 0$	Add. Id (2c)		
	$= -1 \cdot x + x - x$	Add. Id (2c)		
	$= x(-1+1) - x$	Distrib. (2e)		
	$= x(0) - x$	Add. Inv. (2d)		
	$= 0 - x$	Mult. by 0 (6)	$= -x$	Add. Id (2c)

$\therefore x \in \mathbb{R} \Rightarrow (-1)x = -x$

■

9. xy = 0 ⇒ at least x or y = 0

Prove that if x and y are real numbers then $x \cdot y = 0$ then $x = 0$ or $y = 0$.
Let $x, y \in \mathbb{R} : x \cdot y = 0$.

i) If $x \neq 0$:

$$\begin{aligned}
x \cdot y &= 0 \\
x \cdot y &= (x + (-x)) && \text{Add. Id (2c)} \\
\frac{1}{x} \cdot x \cdot y &= \frac{1}{x}(x + (-x)) && \text{Add. Id (2c)} \\
\frac{1}{x} \cdot x \cdot y &= \frac{1}{x}(x + (-x)) && \text{Mult. Inv (2c)} \\
\frac{1}{x} \cdot x \cdot y &= \frac{1}{x}(x) + \frac{1}{x}(-x) && \text{Distrib. (2e)} \\
y &= 1 + (-1) && \text{Mult Inv. (2c)} \\
y &= 0 && \text{Mult Inv. (2c)}
\end{aligned}$$

ii) If both $x, y = 0$ then $x \cdot y = 0 \cdot 0 = 0$ by No. 6.

$\therefore$ If the product of two elements is 0, then one or both elements must equal zero.

2 Order

11. Verifying facts with Order Axioms

(a) a is positive $\Rightarrow$ -a is negative

Let a be a real number. Show that if a is positive, then $-a$ is negative. Conversely, if a is negative, show that $-a$ is positive.

Let $a \in \mathbb{R}^+$. By (10b): $-a$ must be $\in \mathbb{R}^- \cup \{0\}$. For $a \in \mathbb{R}^+$, $a + 0 \neq 0$, so $-a$ can't be $\in \{0\}$. Accordingly, $-a$ must be an element of the last of the disjoint sets, $\mathbb{R}^-$. Thus when a is positive, $-a$ is negative.

Similarly, if $a \in \mathbb{R}^-$, then by the Multiplicative ID (2c), $-a * 1 = -(-a)(1) = a$ by (10b). If $-a \in \{0\}$, then $-a * 1 = 0$ by multiplication by zero (2c), so $-a$ can't be zero. If $-a \in \mathbb{R}^+$, then $-a * 1 = -a \neq a$, so $-a$ can't be $\in \mathbb{R}^+$. Accordingly, if a is negative, $-a$ is positive.

$\blacksquare$

(b) $1 \in \mathbb{R}$ is positive

Show that the real number 1 is positive:
Let $x \in \mathbb{R}$. Then by the multiplicative ID (2c), $1 \cdot x = x$. Since $1 \in \mathbb{R}$, 1 is either positive, negative, or 0 (10b). But since $1 \neq 0$, 1 is either positive or negative. If $1 \in \mathbb{R}^-$, then $1 \cdot x = -x$ because of multiplication by -a (8). This contradicts the multiplicative identity (2c), so 1 cannot be negative. Accordingly, 1 must be an element of the last disjoint set: 1 must be positive.

$\blacksquare$

(c) $\exists$ a negative number

Show that there exists a negative number.
Let $x \in \mathbb{R}^+$. Then by Additive Inverses (2d), exists. Because of disjoint sets of real numbers (10b), $-x$ can not be positive. Because of Multiplicative Identities (2c), $-x$ cannot be zero. Therefore, when x is positive $-x$ is negative, so there exists a negative number.

$\blacksquare$

13. The relation $\leq$ on $\mathbb{R}$

(a) Symmetry

For all real numbers x, $x \leq x$.

$x - x = 0$	Add. Id (2c)
$0 \in \{0\}$	vacuously true
$\{0\} \in \mathbb{R}^+ \cup \{0\}$	
$x - x \in \mathbb{R}^+ \cup \{0\}$	
$x \leq x$	Defn. less than (12)

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(b) Reflexivity

For all real numbers x and y, if $x \leq y$ and $y \leq x$ then $x = y$.

Let c, d $\in \mathbb{R}^+ \cup \{0\}$.	
$x \geq y \Rightarrow x - y = c$	Def. GEQ (12)
$y \geq x \Rightarrow y - x = d$	Def. GEQ (12)
$c + d = x - y + y - x$	substitution
$c + d = 0$	Add. Id (2c)

Then because c and d are each either positive or zero by Order Axiom 10b, we will consider the combinations of each to determine which satisfies the above finding that $c + d = 0$:

1. $c, d \in \mathbb{R}^+$
   By Order Axiom 10a, the positive reals are closed under addition, thus $c + d \in \mathbb{R}^+ \Rightarrow c + d \neq 0$ so $c, d \notin \mathbb{R}^+$.
2. $c \in \mathbb{R}^+$, $d \in \{0\}$
   By the Additive Identity (2c), $c + d = c \in \mathbb{R}^+ \Rightarrow c + d \neq 0$ so $c \notin \mathbb{R}^+$ while $d = 0$.
3. $c, d \in \{0\}$ With c, d both $\in \{0\}$, $c + d = 0 + 0 = 0$.

By substituting the equations found from Def. GEQ (12), $c = x - y = 0$, so $x = y$ by the Additive Identity. WLOG, $d = y - x = 0 \Rightarrow x = y$.

$$\therefore\ x \leq y \text{ and } y \leq x \Rightarrow x = y$$

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(c) Transitivity

For all real numbers x, y, z, if $x \le y$ and $y \le z$, then $x \le z$.

Let c, d $\in \mathbb{R}^+ \cup \{0\}$.	
$x \le y \Rightarrow y - x = c$	
$\Rightarrow y = c + x$	
$y \le z \Rightarrow z - y = d$	
$\Rightarrow y = z - d$	
$c + x = z - d$	y = y
$z - x = c + d$	
$\Rightarrow c + d \in \mathbb{R}^+ \cup \{0\}$	(10a) & Add Id (2c)
$z - x \in \mathbb{R}^+ \cup \{0\}$	
$x \le z$	Def. LEQ (12)

■

14. Operations with Inequalities

(a) a > b and c ≥ d ⇒ a+c > b+d

Let $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^+ \cup \{0\}$.	
$a > b \Rightarrow a - b = x$	Def. greater than (12)
$c \ge d \Rightarrow c - d = y$	Def. GEQ (12)
$x + y = a - b + c - d$	substitution
$x + y = (a + c) - (b + d)$	Distr. (2e)
If $y \in \mathbb{R}^+$, then $x + y \in \mathbb{R}^+$	(10a)
If $y \in \{0\}$, then $x + y \in \mathbb{R}^+$	Add. ID (2c)
$\Rightarrow (a + c) - (b + d) \in \mathbb{R}^+$	equality with $x + y$
$\Rightarrow a + c > b + d$	Def greater than (12)

■

(b) a > b > 0 and c ≥ d >0 ⇒ ac > bd

If $a > b$ and $c \geq d$ then $a + c > b + d$.

Let $x, y \in \mathbb{R}^+$ and $x \in \mathbb{R}^+ \cup \{0\}$.	
$a > b \Rightarrow a - b = x \Rightarrow a = x + b$	(12)
$b > 0 \Rightarrow b - 0 = y \Rightarrow b = y$	(12)
$c \geq d \Rightarrow c - d = z \Rightarrow c = z + d$	(12)
$ac = (x + b)(z + d)$	substitution
$ac = xz + xd + bz + bd$	Distr. (2e)
$ac - bd = xz + xd + bz$	
$ac - bd = xd + z(x + b)$	Distr. (2e)
$x, d \in \mathbb{R}^+ \Rightarrow xd \in \mathbb{R}^+$	(10a)
$z(x + b) \in \mathbb{R}^+ \cup \{0\} \Rightarrow xd + z(x + b) \in \mathbb{R}^+$	(10a) and (2c)
$\Rightarrow ac - bd \in \mathbb{R}^+$	equality with xd + z(x + b)
$\Rightarrow ac > bd$	Def. greater than (12)

■

(c) a > b and c < 0 ⇒ ac < bc

If $a > b$ and $c < 0$ then $ac < bc$.

Let x, y $\in \mathbb{R}^+$	
$a > b \Rightarrow a - b = x$	Def. greater than (12)
$c < 0 \Rightarrow 0 - c = y \Rightarrow c = -y$	Def. greater than (12)
$cx = c(a - b)$	substitution
$ca - cb = cx$	Distr. (2e)
$(-y)a - (-y)b = (-y)x$	substitution of c
$-ya + yb = -yx$	(7) & (11a)
$-1(yb - ya = -yx)$	Comm. (2a) and Mult. by -1
$-yb - (-ya) = yx$	Distr. (2e)
$cb - ca = yx \in \mathbb{R}^+$	substitution of -y
$\Rightarrow yx \in \mathbb{R}^+$	Closure (10a)
$\Rightarrow cb - ca \in \mathbb{R}^+$	equality with yx
$\Rightarrow cb > ca$	Def. greater than (12)

■

15. Negation in Inequalities

$b - a = x \in \mathbb{R}^+$	Defn. less than (12)
$-a + b = x$	Commut. (2a)
$-a + (-1)(-1)b = x$	(11a)
$-a + (-(-b)) = x$	(8)
$(-a) - (-b) = x \in \mathbb{R}^+$	(11a)
$-a > -b$	Defn. greater than

$\therefore$ a < b ⇒ -b < -a

■

16. For a ∈ ℝ, a² ≥ 0

Prove that $a^2 \geq 0$ for any real number a.
By Order Axiom No. 10b:

$$a \in \mathbb{R} \Rightarrow a \in \mathbb{R}^+ \cup a \in \{0\} \cup a \in \mathbb{R}^-$$

Accordingly, it needs to be shown that a in all three disjoint sets is $\in \mathbb{R} \cup \{0\}$:

1. $a \in \mathbb{R}^+$:
   By Order Axiom 10a, a is closed under multiplication.
   So $a^2 = (a)(a) \in \mathbb{R}^+ \Rightarrow a^2 \in \mathbb{R} \cup \{0\}$.

2. $a \in \{0\}$:
   By No. 6, $x(0) = 0$.
   So $a^2 = (a)(a) = (0)(0) = 0 \in \{0\} \Rightarrow a^2 \in \mathbb{R} \cup \{0\}$

3. $a \in \mathbb{R}^-$:
   By 11a, if a is negative, then -a is positive.
   Let a = -b, where b $\in \mathbb{R}^+$.
   So, $a^2 = (a)(a) = (a)(-b)$, which by No. 7 $= (-a)(b)$.
   Then because of 11a, $-a$ must $= c \in \mathbb{R}^+$.
   Accordingly, $(a)(a) = (c)(b) \in \mathbb{R}^+$ because both b and c are $\in \mathbb{R}^+$, and thus are closed under multiplication by Order Axiom 10a.
   So, $a^2 \in \mathbb{R} \cup \{0\}$.

$\therefore$ $a^2 \geq 0$ for any real number a.

■

3 Absolute Value

19. $|a| = |-a|$

Prove that if a is a real number then $|a| = |-a|$.
First consider the case that $a = 0$. Then $|a| = |0| = 0$ by No. 18 and $|-a| = |(-1)(0)| = |0| = 0$ by No. 8 (multiplication by -1), No. 6 (multiplication by zero), and No. 18 (Def. abs. val.). So when $a = 0, 0 = |a| = |-a|$ so the proof is satisfied. By Order Axiom 10b, if a is not zero, then a can only be positive or negative:

Let $a \in \mathbb{R}^+$	
$\Rightarrow a \geq 0\,|a| = a$	Def. Abs. Val (18)
If $a \in \mathbb{R}^+$, then $-a \in \mathbb{R}^-$	(11a)
$\Rightarrow -a < 0 \Rightarrow |-a| = -(-a)$	Def. Abs. Val. (18)
$|-a| = (-1)(-1)(a) = a$	(11a)

WLOG, beginning with $a \in \mathbb{R}^-$ produces the same result.
$\therefore\ |a| = a = |-a|$ so $|a| = |-a|$.

■

20. Properties of Abs. Val

a) Addition

Show: $|a+b| \leq |a| + |b|$

i) Let $a, b \in \mathbb{R}^+ \cup \{0\}$

$a + b \in \mathbb{R} + \cup \{0\}$	closure (10a)
$\|a+b\| = a + b$	Def. Abs. Val (18)
$\|a\| = a$	Def. Abs. Val (18)
$\|b\| = b$	Def. Abs. Val (18)
$\Rightarrow \|a\| + \|b\| = a + b$	substitution
$\|a+b\| = ab = \|a\| + \|b\|$	substitution
$\|a+b\| \leq \|a\| + \|b\|$	

ii) Let $-a \in \mathbb{R}^+ \cup \{0\}$ and $b \in \mathbb{R}^-$.
(WLOG, this case pertains to $-b \in \mathbb{R}^+ \cup \{0\}$ and $a \in \mathbb{R}^-$).

If $-a + b \geq 0$, $|-a+b| = b - a$:

NTS: $\|-a\| + \|b\| - \|-a+b\| \in \mathbb{R}^+$.	
$\|-a\| + \|b\| - (b-a)$	substitution
$\|-a\| + \|b\| - b + a$	Mult. by -1 (11a)
$-(-a) + b - b + a$	Def. Abs. Val. (18)
$a + a$	Add. Id (2c)
$2a, \in \mathbb{R}^+$	(11a)

If $-a + b < 0$, $|-a+b| = -(-a+b) = a - b$:

NTS: $\|-a\| + \|b\| - \|-a+b\| \in \mathbb{R}^+$.	
$\|-a\| + \|b\| - (a-b)$	substitution
$\|-a\| + \|b\| - a + b$	Mult. by -1 (11a)
$-(-a) + b - a + b$	Def. Abs. Val. (18)
$a + b - a + b$	Add. Id (2c)
$2b \in \mathbb{R}^+$	(11a)

iii) Let $-a, -b \in \mathbb{R}^-$

$-a + -b \in \mathbb{R}-$	closure (10a)
$\lvert -a + -b \rvert = -(-a - b)$	Def. Abs. Val (18)
$\lvert -a + -b \rvert = a + b$	(11a)
$\lvert -a \rvert = -(-a) = a$	Def. Abs. Val (18), (11a)
$\lvert -b \rvert = -(-b) = b$	Def. Abs. Val (18), (11a)
$\Rightarrow \ \lvert a \rvert + \lvert b \rvert = a + b$	substitution
$\lvert -a + -b \rvert = ab = \lvert -a \rvert + \lvert -b \rvert$	substitution
$\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$	

b) Multiplication

Show: $|a \cdot b| = |a| \cdot |b|$

i) Let $a, b \in \mathbb{R}^+ \cup \{0\}$

$\lvert a \cdot b \rvert = ab$	Def. Abs. Val (18)
$\lvert a \rvert = a$	Def. Abs. Val (18)
$\lvert b \rvert = b$	Def. Abs. Val (18)
$\Rightarrow \ \lvert a \rvert \cdot \lvert b \rvert = ab$	substitution
$\lvert a \cdot b \rvert = ab = \lvert a \rvert \cdot \lvert b \rvert$	substitution

ii) Let $a \in \mathbb{R}^+ \cup \{0\}$ and $b \in \mathbb{R}^-$.
(WLOG, this case pertains to $b \in \mathbb{R}^+ \cup \{0\}$ and $a \in \mathbb{R}^-$).

$a \cdot b \in \mathbb{R}^-$	(11a)
$\lvert a \cdot b \rvert = -ab$	Def. Abs. Val (18)
$\lvert a \rvert = a$	Def. Abs. Val (18)
$\lvert b \rvert = -b$	Def. Abs. Val (18)
$\Rightarrow \ \lvert a \rvert \cdot \lvert b \rvert = a(-b) = -ab$	substitution & (7)
$\lvert a \rvert = -ab = \lvert a \rvert \cdot \lvert b \rvert$	substitution

iii) Let $a, b \in \mathbb{R}^-$

$a \cdot b \in \mathbb{R}^+$	(11a)
$\lvert a \cdot b\rvert = ab$	Def. Abs. Val (18)
$\lvert a\rvert = -a$	Def. Abs. Val (18)
$\lvert b\rvert = -b$	Def. Abs. Val (18)
$\Rightarrow \lvert a\rvert \cdot \lvert b\rvert = (-a)(-b) = (-1)(-1)(ab) = ab$	(2a), (11a)
$\lvert a \cdot b\rvert = ab = \lvert a\rvert \cdot \lvert b\rvert$	substitution

c) Subtraction

Show: $|a - b| \geq ||a| - |b||$
Note: By 19, $|a - b| = | - (a - b)| = |b - a|$
NTS: both $|a| - |b|$ and $|b| - |a|$ are less than or equal to $|a - b| = |b - a|$.

i) When $|a| - |b| \geq 0$, $||a| - |b|| = |a| - |b|$.

$\lvert a\rvert = \lvert a\rvert$	
$\lvert a\rvert = \lvert a + 0\rvert$	Add. Id (2c)
$\lvert a\rvert = \lvert a + b - b\rvert$	Add. Id (2c)
$\lvert a\rvert = \lvert (a - b) + b\rvert$	Add. Id (2c)
$\lvert a\rvert \leq \lvert a - b\rvert + \lvert b\rvert$	Add. Abs. Val (20a)
$\lvert a\rvert - \lvert b\rvert \leq \lvert a - b\rvert$	Add. Id (2c)

ii) When $|a| - |b| \leq 0$ $||a| - |b|| = |b| - |a|$.

$\lvert b\rvert = \lvert b\rvert$	
$\lvert b\rvert = \lvert b + 0\rvert$	Add. Id (2c)
$\lvert b\rvert = \lvert b + a - a\rvert$	Add. Id (2c)
$\lvert b\rvert = \lvert (b - a) + a\rvert$	Add. Id (2c)
$\lvert b\rvert \leq \lvert b - a\rvert + \lvert a\rvert$	Add. Abs. Val (20a)
$\lvert b\rvert - \lvert a\rvert \leq \lvert b - a\rvert$	Add. Id (2c)

$\therefore$ $|a - b| \geq ||a| - |b||$

■

21. $\epsilon > 0 \Rightarrow |a| < \epsilon \iff -\epsilon < a < \epsilon$

Prove that if $\epsilon > 0$ then $|a| < \epsilon$ if and only if $-\epsilon < a < \epsilon$.

1. NTS: $|a| < \epsilon \Rightarrow -\epsilon < a < \epsilon$

i) $|a| \geq 0 \Rightarrow |a| = a$	Def. Abs. Val (18)
$|a| < \epsilon \Rightarrow a < \epsilon$	substitution
ii) $|a| < 0 \Rightarrow |a| = -a$	Def. Abs. Val (18)
$|a| < \epsilon \Rightarrow -a < \epsilon$	substitution
$\Rightarrow -1(-a) > -1(\epsilon)$	(15)
$a > -\epsilon$	(11)

Thus combining the two inequalities attained from (i) and (ii) gives: $-\epsilon < a < \epsilon$

2. NTS: $-\epsilon < a < \epsilon \Rightarrow |a| < \epsilon$

i) $a > -\epsilon$	
$a - (-\epsilon) \in \mathbb{R}^+$	Def. Greater Than (12)
$-(-a) + \epsilon \in \mathbb{R}^+$	(11)
$\epsilon - (-a) \in \mathbb{R}^+$	Comm. (2a)
$-a < \epsilon$	Def. Less Than (12)
ii) $a < \epsilon$	given as right side of inequality

Thus because $|a| = -a$ or a by Def. Abs. Val (18), and both $-a$ an a have been shown to be less than ϵ, $|a| < \epsilon$.

$\therefore\ \epsilon > 0 \Rightarrow |a| < \epsilon \iff -\epsilon < a < \epsilon$

■

22. Satisfying Abs. Val. Inequalities

Find the values of x that satisfy the following inequalities.

a) $|1 - x| < 4$

$-4 < 1 - x < 4$	Abs. Val. Ineq. (21)
$-4 + (-1) < 1 - x + (-1) < 4 + (-1)$	addition of -1
$-5 < -x < 3$	Add. ID (3)
$(-1)(-5) > (-1)(-x) > (-1)(3)$	multiplication by -1 (15)
$5 > x > -3$	(11a)

$\therefore \{x \in \mathbb{R} : -3 < x < 5\}$.

b) $|x^2 - x - 1| < x^2$

$-x^2 < x^2 - x - 1 < x^2$	Abs. Val. Ineq (21)
$-x^2 + (-x^2) < x^2 - x - 1 + (-x^2) < x^2 + (-x^2)$	
$-2x^2 < -x - 1 < 0$	Add. Id (2c) / Comm. (2a)
$-2x^2 + 1 < -x - 1 + 1 < 0 + 1$	add +1
$-2x^2 + 1 < -x < 1$	Add. Id (2c)
$-1(-2x^2 + 1) < -1(-x) < -1(1)$	Mult. by -1
$2x^2 - 1 > x > -1$	Distr. (2e), 8, 11a

From the right side we can see that $x > -1$.
Now consider the left:

$2x^2 - 1 > x$	
$2x^2 - x - 1 > 0$	Add. Id (2c)
$(2x + 1)(x - 1) > 0$	Distr. (2e)

If $(2x + 1)(x - 1)$ is going to be greater than 0, then by 10a either (i) or

(ii) most occur:

i) both factors must $\in \mathbb{R}^+$:

$(2x+1) > 0 \Rightarrow x > -\dfrac{1}{2}$

$(x-1) > 0 \Rightarrow x > 1$

ii) both must be $\in \mathbb{R}^-$:

$(2x+1) < 0 \Rightarrow x < -\dfrac{1}{2}$

$(x-1) < 0 \Rightarrow x < 1$

Because $x > -\frac{1}{2}$ together with $x > 1$ gives $x > 1$, and $x < -\frac{1}{2}$ together with $x < 1$ gives $x < -\frac{1}{2}$, the final constraints on x are $x > -1, x < -\frac{1}{2}$ and $x > 1$.

$\therefore \{x \in \mathbb{R} : -1 < x < -\frac{1}{2}, x \geq 1\}$.

■

23. $|x| \leq 1 \Rightarrow |x^2 - 1| \leq 2|x-1|$

$\lvert x\rvert \leq 1 \Rightarrow -1 \leq x \leq 1$	No. 21
$-1+1 \leq x+1 \leq 1+1$	Addition of 1
$0 \leq x+1 \leq 2$	simplify
$x+1 \geq 0 \Rightarrow \lvert x+1\rvert = x+1$	Def. Abs. Val. (18)
$0 \leq \lvert x+1\rvert \leq 2$	substitution
On the left this gives $-1 \leq x \leq 1$ which is given.	
On the right, consider $\lvert x+1\rvert \leq 2$:	
$\lvert x+1\rvert(1) \leq 2(1)$	Mult. Id (2c)
$\lvert x+1\rvert\lvert x-1\rvert\lvert\frac{1}{\lvert x-1\rvert}\rvert \leq 2\lvert x-1\rvert\lvert\frac{1}{\lvert x-1\rvert}\rvert$	Mult. Inv. (2d)
$\lvert x+1\rvert\lvert x-1\rvert \leq 2\lvert x-1\rvert$	Cancellation
$\lvert(x+1)(x-1)\rvert \leq 2\lvert x-1\rvert$	(20b)
$\lvert x^2-1\rvert \leq 2\lvert x-1\rvert$	Distr. (2e)

$\therefore |x| \leq 1 \Rightarrow |x^2 - 1| \leq 2|x-1|$

■

4 Limit Points

27. Example of No. 29

The statement - The point 6 is not a limit point of the interval (6, 9) - is false.

By No. 29, a is a limit point of (a, b). Accordingly, 6 is a limit point of $(6, 9)$.

28. Element of open interval an L.P.

Let $p \in (a, b)$. Prove p is a limit point of (a, b).

By the definition of an open interval, $p \in (a, b) \Rightarrow a < p < b$. It needs to be shown that there exists some m distinct from p that is also in this interval (a, b). Such an m will be either less than p or greater than p. WLOG, let us consider the case that m is less than p. To show that p is a limit point of (a, b), we need to show that every interval containing p also contains m. Let p_1 be the lower bound of the open interval containing p. This lower bound of the interval containing p will be either less than a or greater than a. First, for $p_1 < a$:

$$\begin{aligned} a < p \Rightarrow p - a = \delta \in \mathbb{R}^+ \\ a = p - \delta \\ p - \delta < p - \frac{\delta}{2} < p \\ \text{Let } m = p - \frac{\delta}{2} \\ a < m < p \\ p_1 < p < m < a < b \end{aligned}$$

Accordingly, there exists m both in the given interval (a, b) and open intervals of p when the lower bound of the open interval containing p is less than a.

If instead p_1 is greater than a, then a similar positive δ can be found between p and some $p_1 < p$. So for $p_1 > a$:

$$\begin{aligned} p < p_1 \Rightarrow p_1 - p = \delta \in \mathbb{R}^+ \\ p = p_1 - \delta \\ p_1 - \delta < p - \frac{\delta}{2} < p \\ Let m = p - \frac{\delta}{2} \\ p_1 < m < p \\ a < p_1 < m < p < b \end{aligned}$$

Accordingly, there exists m both in the given interval (a, b) and open intervals of p when the lower bound of the open interval containing p is greater than a.

As an equivalent procedure can be shown for $m > p$, this accounts for all necessary cases of p bounded by real elements inside of or outside of the given interval (a, b). For all open intervals of p, there exists m, $m \neq p$ such that m is an element of both the open interval containing p and the given interval (a, b).

$\therefore p \in (a, b) \Rightarrow p$ is a limit point of (a, b).

■

29. Bound of open interval

Prove that a is a limit point of (a, b).

NTS: Any open interval that contains a must also contain m distinct from a such that m is in both the open interval containing a and the open interval (a, b).

For $\delta, \epsilon \in \mathbb{R}^+$, let $(a-\delta, a+\epsilon)$ represent any open interval containing a. Because the open interval (a, b), only concerns values $> a$, we need only consider the upper half of open intervals containing a, or the interval $(a, a+\epsilon)$.

If $a+\epsilon \geq b$, there must exist m in the interval (a, b), so we consider the a subset (a, b) of the larger interval $(a, a+\epsilon)$, as what is true for the sub-interval (a, b) regarding the existence of m will also be true for $(a, a+\epsilon)$ when $a+\epsilon \geq b$.

$$\begin{aligned} a < b \Rightarrow b - a = \gamma &\in \mathbb{R}^+ \\ a &= b - \gamma \\ b - \gamma < b - \frac{\gamma}{2} &< b \\ \text{Let } m = b - \frac{\gamma}{2} & \\ a - \delta < a < m < b &< a + \epsilon \end{aligned}$$

Accordingly, if the upper bound of the interval containing a is greater than the upper bound of the given set, then there exists m distinct from a both the interval containing p and the given interval (a, b). If $a+\epsilon < b$, then we instead consider γ between a and $a+\epsilon$:

$$\begin{aligned} a < a + \epsilon &< b \\ a < a + \epsilon \Rightarrow a + \epsilon - a = \gamma &\in \mathbb{R}^+ \\ a = (a - \epsilon) &- \gamma \\ (a-\epsilon) - \gamma < (a-\epsilon) - \frac{\gamma}{2} &< a - \epsilon \\ \text{Let } m = (a-\epsilon) - \frac{\gamma}{2} & \\ a - \delta < a < m < a + \epsilon &< b \end{aligned}$$

Accordingly, if the upper bound of the interval containing a is less than the upper bound of the given set, then there exists m distinct from a both the interval containing p and the given interval (a, b).

Thus for the given set (a, b) whether the upper bound of the open interval containing p is less than or greater than b, there will always be $m, m \neq a$ both in (a, b) and $(a-\delta, a+\epsilon)$.

$\therefore$ a is a limit point of (a, b).

■

30. Non-element of closed interval

Suppose $p \notin [a, b]$. Prove p is not a limit point of (a, b).

Because $p \notin [a, b]$, either $p < a$ or $p > b$. WLOG, let us consider the case that $p < a$:

$$p < a \Rightarrow a - p = d, d \in \mathbb{R}^+$$

Accordingly, $a = p + d$, and we need to show that there exists a point in an open interval containing p but not also in the open interval $[a, b]$. Consider the open interval $M = (p - d, p + d)$ such that $p \in M$:

$$p - d < p - \frac{d}{2} < p + d$$

Thus $p - \frac{d}{2} \in M$, but $p - \frac{d}{2} \notin [a, b]$.

So for $(p - d, p + d)$ containing p, $\exists \ m \in (p - d, p + d) \ : \ m \notin [a, b]$. Thus for $p < a$, $\exists$ an open interval containing p which contains no element of $[a, b]$, so p can not be a limit point of $[a, b]$.

Similarly, if $p > b$ then $p - b = g$, $g \in \mathbb{R}^+$. So for $M = (p - g, p + g)$, $p \in M$ and $(p - \frac{g}{2}) \in M$, but $(p - \frac{g}{2}) \notin [a, b]$. Thus for $p > b$, $\exists$ an interval containing p which contains no element of $[a, b]$, so p can not be a limit point of $[a, b]$.

Thus when $p < a$ and when $p > b$, p is not a limit point of $[a, b]$.
$\therefore \ p \notin [a, b] \ \Rightarrow p$ is not a limit point of (a, b).

■

31. M with a L.P. has 2 elements and is infinite

Suppose M is a point set that contains a limit point. Prove that M has at least two points.

Because $p \in M$ and p is a limit point, for all open intervals (a, b) containing p, there must exist another point, $m \in (a, b) : m \neq p$ so that m is also in M. Thus because p and m are both in M, M contains at least two points.

■

Accordingly because p is a limit point in M, for all open intervals (a, b) containing p there must exist two distinct points in M.

Consider the interval $(a, b)\backslash m$. As shown above, there must exist some other $m_1 \in M : m_1 \neq m, m_1 \neq p$.

Again if we consider $(a, b)\backslash\{m, m_1\}$, because M contains p and thus must contain at least 2 distinct points as shown above, there must exist another point $m_2 \in M$.

Because the existence of $m_i : m_i \neq p$ must be true for all open intervals containing p, this process can be repeated infinitely many times. Thus there must be an infinite number of m_i's in M.

32. Limit Point of $2 + \frac{1}{n}$, $n \in \mathbb{N}$

Prove that 2 is a limit point of $M = \{2 + \frac{1}{n} : n \in \mathbb{N}\}$

For δ, $\epsilon \in \mathbb{R}^+$ consider the open interval $(2 - \delta, 2 + \epsilon)$. For all such intervals, there exists $\frac{1}{n} > \delta$ and $\frac{1}{n} < \epsilon$ so the set S described below is a fair representation of all open intervals containing 2.

$$S = (2 - \frac{1}{n}, 2 + \frac{1}{n})$$

$$2 - \frac{1}{n} < 2 < 2 + \frac{1}{2n} < 2 + \frac{1}{n}$$

Let $m = 2 + \frac{1}{2n}$. So we have $m \neq 2$ such that $m \in M$ and $m \in S$.

$\therefore$ 2 is a limit point of $M = \{2 + \frac{1}{n} : n \in \mathbb{N}\}$

■

33. Nonexistence of limit point of discrete finite set

Unbridgability: For $m_1, m_2, m_x \in M$, $m_1 \neq m_2$, if $\forall\ m_x\, in\, M \nexists\ m_x$ in the open interval (m_1, m_2), then there exists an open interval which contains no elements of M. If such an interval exists, then for $p \in \mathbb{R}$, every open interval containing p contains a smaller open interval which will not include an element of M distinct from p, so M can have no limit points.

Let $M = \{1, 2, 3\}$. Prove that M has no limit points.

The open interval $(1, 2)$ contains neither 1, 2, nor 3 (by the definition of an open interval and greater than). Thus there exists an open interval in M such that $\forall\ m_x \in M \nexists m_x \in (1, 2)$. Thus because of unbridgability, M has no limit points.

34. $\mathbb{Z}$ has no friends

Prove that $\mathbb{Z}$ has no limit points.

FSOC, suppose p is a limit point of the integers. Then there can be no open interval containing p which does not also include an integer distinct from p. Thus if p is not a limit point of $\mathbb{Z}$, it must be shown that for all possible p's there exists an an open interval containing p with no other integers besides p.

1. $p \in \mathbb{Z}$
   If p is itself an integer, then consider the open interval $(p-1, p+1)$. Because the distance between two consecutive integers is 1, the only integers in this interval is p. The same is true for any smaller open interval containing p. Any larger open interval containing p must include such a smaller open interval, so *for all* open intervals containing $p \in \mathbb{Z}$: $\nexists \ z \in \mathbb{Z} \ : \ p \neq z$. Accordingly by the definition of a limit point, $p \in \mathbb{Z}$ cannot not be a limit point of $\mathbb{Z}$.

2. $p \notin \mathbb{Z}$
   If p is not an integer, then let $z_1 = \lfloor p \rfloor$, $z_2 = \lceil p \rceil$, and consider the open interval (z_1, z_2). Because the distance between z_1 and z_2 is 1 and z_1 and z_2 are not themselves in the open interval (z_1, z_2), there are no integers in the open interval (z_1, z_2). The same is true for any smaller open interval containing p. Any larger open interval containing p must include such a smaller open interval, so *for all* open intervals containing $p \notin \mathbb{Z} : \nexists \ z \in \mathbb{Z}$. Accordingly by the definition of a limit point, $p \notin \mathbb{Z}$ cannot not be a limit point of $\mathbb{Z}$.

Thus for neither $p \in \mathbb{Z}$ nor $p \notin \mathbb{Z}$ can p be a limit point of $\mathbb{Z}$.

$\therefore$ $\mathbb{Z}$ has no limit points.

$\blacksquare$

35. Implications of $H \cap K$

Suppose H, K are nonempty point sets. Assume p is a limit point of the set $H \cap K$. Prove that p is a limit point of H and p is a limit point of K.

p is a limit point of $H \cap K \Rightarrow$
$\forall$ intervals containing p, $\exists\, m \in H \cap K : m \neq p$

Because, $(H \cap K) \subset H$ and $(H \cap K) \subset K$,
$m \in H \cap K \Rightarrow m \in H\, and\, m \in K.$

Accordingly, the necessary m exists in H and K considered separately, so p is a limit point of H and p is a limit point of K.

■

36. Implications of $H \cup K$

Let H, K be nonempty point sets. Then the statement "If p is a limit point of $H \cup K$, then p is a limit point of H and p is a limit point of K" is false.

FSOC, assume the statement is true, that p a limit point of $H \cup K$ implies that p is a limit point of both H and K alone.

Let H be the open interval $(1, 2)$, and K the open interval $(3, 4)$, where 2 is a limit point of $H \cup K$. Accordingly, by No. 29, 2 is a limit point of H because in all intervals containing 2, there can be found another element in H that is not equal to 2. But between 2 and the lower bound of K there is a positive real distance of 1. Thus for all intervals containing 2, i.e. the open interval (1.5, 2.5), there does not necessarily exist another element of the interval such that the element is also in K. So while 2 is a limit point of H, it is not also a limit point of K.

Thus by counter example, the statement must be false.

■

5 Supremum, Infimum

42. Min, Max, Inf, Sup

What are the min, max, infimum and supremum of the set $M_1 = \{-2, 1.31, 7, 10\}$, and $M_2 = (-2, 9)$

$\forall\ m \in\ M_1$, $-2\ \leq\ m$, so -2 is the min of M_1. Furthermore, every number in M_1 greater than 2 is not a lower bound of M, so 2 is the greatest lower bound and thus the infimum of M_1.

$\forall\ m \in\ M$, $10\ \geq\ m$, 10 is the max of M_1. Furthermore, every number in M_1 less than 10 is not an upper bound of M_1, so 10 is the least upper bound of M_1 and thus the supremeum of M_1.

The open interval $M_2 = (-2, 9)$ is infinitely large and does not have a min or max. Rather, $\forall\ m\ \in\ M_2$, $-2\ <\ m$, so -2 is the infimum. Similarly, $\forall\ m \in\ M_2$, $9 > m$, so 9 in the supremum.

43. Infimum of $M = \{3 + \frac{1}{n}\}$

Prove the infimum of the set $M = \{3 + \frac{1}{n} : n \in \mathbb{N}\}$ is 3.

Let $x \in M$. So $x = 3 + \frac{1}{n}$.

$$x - 3 = (3 + \frac{1}{n}) - 3 = \frac{1}{n} = d \in\ \mathbb{R}^+$$

Accordingly, $3 \leq x\ \ \forall\ \ x \in M$, so 3 is a lower bound of M.

Furthermore, for $q \in M$, let $q = 3 + \frac{1}{2n}$.

$$3 < 3 + \frac{1}{2n} < 3 + \frac{1}{n}$$

Accordingly, $\exists\ \ q \in M$ such that $3 < q < x\ \ \forall\ \ x \in M$, so x is not a lower bound for all $x > 3$. Thus 3 must be the greatest lower bound of x.

$\therefore$ For $M = \{3 + \frac{1}{n} : n \in \mathbb{N}\}$, $infM = 3$.

■

44. Supremum of $\frac{1}{t}$, $t \in (2, \infty)$

What is the supremum of the set $M = \{\frac{1}{t} : t \in (2, \infty)\}$? Prove that your answer is correct.

$$t \in (2, \infty) \Rightarrow 2 < t < \infty$$
$$\frac{2}{t} < \frac{t}{t}$$
$$\frac{1}{t} < \frac{1}{2}$$

Thus $t < \frac{1}{2}\ \forall\ t \in (2, \infty)$, so $\frac{1}{2}$ is an upper bound of M.

For $\epsilon \in \mathbb{R}^+$, let $S' = 2 + \epsilon : S' \in M$. Let $S'' = \frac{1}{2+(\epsilon \backslash 2)}$.

$$2 < 2 + \frac{\epsilon}{2} < 2+$$

$$\frac{1}{2+} < \frac{1}{2+(\epsilon \backslash 2)} < \frac{1}{2}$$

$$S' < S'' < supM$$

Thus for every $S' < \frac{1}{2}$, $\exists S'' : S'' > S'$.
So for every $S' < \frac{1}{2}$, S' is not an upper bound of M, so $\frac{1}{2}$ is the *least* upper bound.

$\therefore$ Because $\frac{1}{2}$ is the least upper bound of M, $\frac{1}{2}$ is the supremum of M

$\blacksquare$

45. $\exists\ a \in M\ :\ \sup M - \epsilon < a \leq \sup M$

Show that if M has a supremum then for any $\epsilon > 0$ there exists $a \in M$ such that $\sup M - \epsilon < a \leq \sup M$.

Let $\sup M = S$.

Case 1: $S \in M$. If $S \in M$, then let $a = S$. Thus $a = \sup M$.

Case 2: $S \notin M$. FSOC, Suppose $\nexists\ a$ such that $S - \epsilon < a < S$.

$\Rightarrow\ \nexists\ S' \in M$ such that $S - \epsilon < S' < S$

$\Rightarrow\ S - \epsilon$ is the greatest upper bound

$\Rightarrow\ S - \epsilon = S\ \ \rightarrow\leftarrow$

Thus $\exists\ a$ such that $\sup M - \epsilon < a < supM$.

Accordingly, when $\sup M \in M$, $\exists\ a\ :\ a = \sup M$ and when $\sup M \notin M$, $\exists\ a$ such that $\sup M - \epsilon < a < \sup M$.

$\therefore$ If M has a supremum then for any $\epsilon > 0$ there exists $a \in M$ such that $\sup M - \epsilon < a \leq \sup M$.

$\blacksquare$

46. Multiplication by a positive number

Theorem. *If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0$, then there exists a positive integer n such that $nx > y$.*

Taylor's Lemma: The natural numbers are not bounded from above.

FSOC, suppose that $\mathbb{N}$ is bounded above. Then by the LUBP, the supremum of $\mathbb{N}$ exists, call it α.

Consider $\alpha - 1$. This cannot be an upper bound because $\alpha - 1$ is less then α and α is the least upper bound. So there exists $n \in \mathbb{N} : n > \alpha - 1$, which further implies $n + 1 > \alpha$. This is a contradiction because $n + 1 \in \mathbb{N}$ and $n + 1 > \alpha$, but we assumed that for all $n \in \mathbb{N}$, $n \leq \alpha$.

Accordingly, $\mathbb{N}$ is not bounded above.

Let $z = \frac{y}{x}$ for $x > 0$. FSOC, suppose there exists $z \in \mathbb{R} : n \leq z$. Then z would be an upper bound for $\mathbb{N}$, but by Taylor's Lemma this cannot be so. Accordingly, for all $z \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > z$.

$$n > z \Rightarrow n > \frac{y}{x} \Rightarrow nx > y$$

$\therefore$ For $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x > 0 \Rightarrow \exists\ n : nx > y$.

■

48. The Bounded Sum of Monotonically Increasing and Decreasing Unbounded and Bounded Sequences

If a bounded sequence is the sum of a monotone increasing and monotone decreasing sequence ($x_n = y_n + z_n$ where (y_n) is monotone increasing and (z_n) is monotone decreasing) (1) does it follow that the sequence converges? (2) What if (y_n) and (z_n) are bounded?

(1) If y_n and z_n are not bounded, then x_n does not necessarily converge. For sake of contradiction, suppose x_n does converge.

Let y_n be the unbounded monotonically increasing sequence $10n + (-1)^n$.

Let z_n be the unbounded monotonically decreasing sequence $-10n+(-1)^n$. Then $x_n = y_n + z_n = 10n + (-1)^n + -10n + (-1)^n = (-1)^n$. So $x_n = (-1)^n$ is a bounded sequence, but it does not converge (by No. 60).

$\therefore$ x_n does not converge when (y_n) and (z_n) are not bounded.

$\blacksquare$

(2) If instead (y_n) and (z_n) are bounded, x_n does converge:

By the Least Upper Bound Property, the supremum of y_n, $\sup y$ exists, and the infimum of z_n, $\inf z$ exists. Let $L = \sup y + \inf z$.
Need to show:

$$\exists\ N\ :\ \forall\ \epsilon > 0,\ |x_n - L|\ <\ \epsilon,\ \forall\ n\ \geq\ N$$

$$\text{where } |x_n - L| = |y_n + z_n - (\sup y + \inf z)|$$

Because (y_n) is increasing and bounded, by 62 it converges. So for every ϵ, particularly $\frac{\epsilon}{2}$, it must be true that $\exists\ N_y\ :\ |y_n - L_y| < \frac{\epsilon}{2}\ \forall n\ \geq\ N_y$. Let $L_y = \sup y$. Similarly, $\exists\ N_z\ :\ |z_n - L_z| < \frac{\epsilon}{2},\ \forall n \geq N$. Let $L_z = \inf z$. Thus,

$$|y_n - \sup Y| < \frac{\epsilon}{2} \text{ and } |z_n - \inf z| < \frac{\epsilon}{2}$$

$$\begin{aligned} |y_n - \sup y| + |z_n - \inf z| &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ |y_n - \sup y + z_n - \inf z| \leq |y_n + \sup y| + |z_n + \inf z| &< \epsilon \\ |y_n - \sup y + z_n - \inf z| &< \epsilon \\ |y_n + z_n - (\sup y + \inf z)| &< \epsilon \\ |x_n - L| &< \epsilon \end{aligned}$$

Let $N_{max} = \max(N_y, N_z)$, Accordingly, $\exists N = N_{max}$ such that

$$\forall\ \epsilon\ >\ 0,\ |x_n - L|\ < \epsilon,\ \forall\ n \geq\ N_{max}$$

$\therefore$, x_n converges when y_n and z_n are bounded.

$\blacksquare$

49. $\sup(A \cup B) \geq \sup A$ and $\sup(A \cap B) \leq \sup A$

Prove that given two sets A and B, then $\sup(A \cup B) \geq \sup A$ and $\sup(A \cap B) \leq \sup A$. Give examples of sets A and B such that the inequalities are strict.

Regarding the union of two sets:

If $\sup B = \sup A$, then $\sup(A \cup B) = \sup A$

If $\sup B > \sup A$, then $\sup(A \cup B) = \sup B > \sup A$

If $\sup B < \sup A$, then $\sup(A \cup B) = \sup A$

Accordingly, the supremum of the union of A and B will be greater than or equal to $\sup A$.

Regarding the intersection of two sets:

If $\sup B = \sup A$, then $\sup(A \cap B) = \sup A$

If $\sup B > \sup A$, then $\sup(A \cap B) = \sup A$

If $\sup B < \sup A$, then $\sup(A \cap B) = \sup B < \sup A$

Accordingly, the supremum of the intersection of A and B will be greater than or equal to $\sup A$.

■

Examples where the inequalities are strict:

$A = \{3, 4, 8, 12\}$ and $B = \{2, 4, 8, 20\}$ Thus $\sup A = 12$ and $\sup B = \{20\}$.

$A \cup B = \{2, 3, 4, 8, 12, 20\} \Rightarrow \ \sup(A \cup B) = 20 > \sup A$

$A \cap B = \{4, 8\} \Rightarrow \ \sup(A \cap B) = 8 < \sup A$

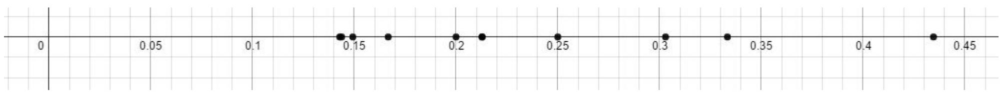
6 Convergence - The Heart of the Matter

53. Graphs with and without Limits

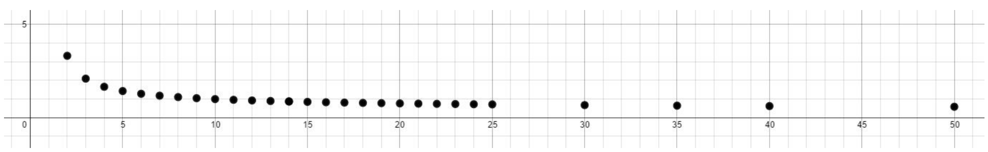
Graph each of the following sequences in 1D along the real number line, and in 2D as functions from $\mathbb{N}$ to $\mathbb{R}$, indicating limits where they exist.

(a) $\left(\frac{1}{\log n}\right)_{n=2}^{\infty}$. Limit = 0.

1D number line



2D graph



Proof: Does converge.
Let $\epsilon > 0$. Let $L = 0$. Let $N = \lceil e^{(1/\epsilon)} \rceil$. Let $n > N$.

$$|\frac{1}{\log N - 0}| = |\frac{1}{\log N}|$$

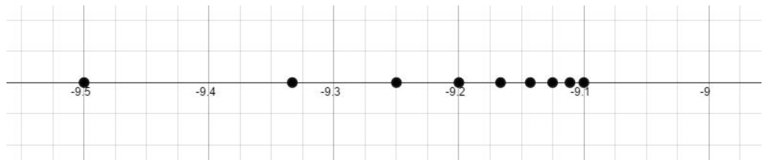
$$\frac{1}{\log N} = \frac{1}{\log \lceil e^{(1/\epsilon)} \rceil} \leq \frac{1}{\log e^{(1/\epsilon)}} = \frac{1}{1/\epsilon} = \epsilon$$

Accordingly, $\forall\ \epsilon > 0,\ \exists\ N :\ \forall\ \epsilon > 0,\ |a_n - L| < \epsilon$, so the sequence converges to 0.

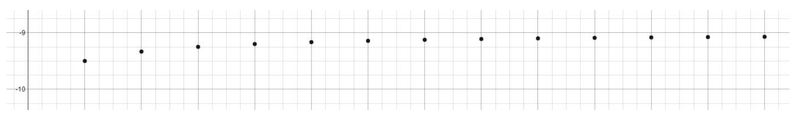
■

(b) $(-10 + \frac{n}{n+1})_{n=1}^{\infty}$. Limit = -9.

1D number line

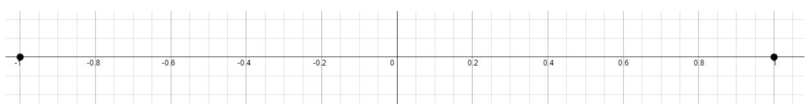


2D graph

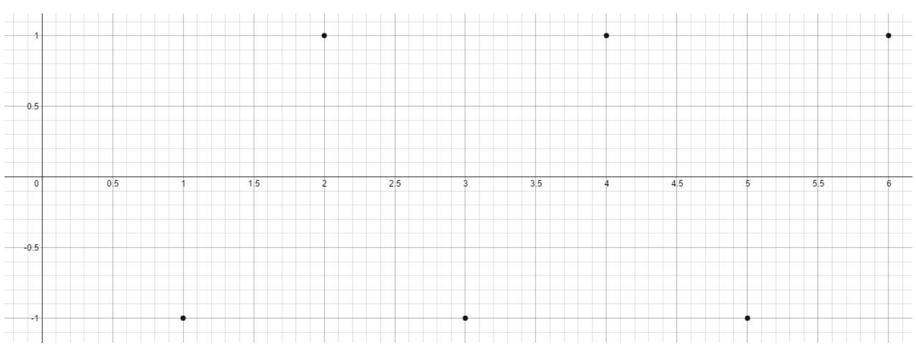


(c) $\left((-1)^n\right)_{n=1}^{\infty}$. Limit DNE.

1D number line

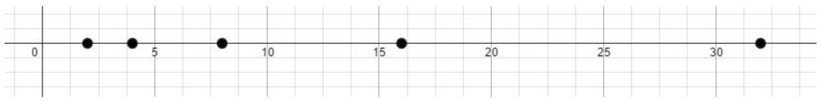


2D graph

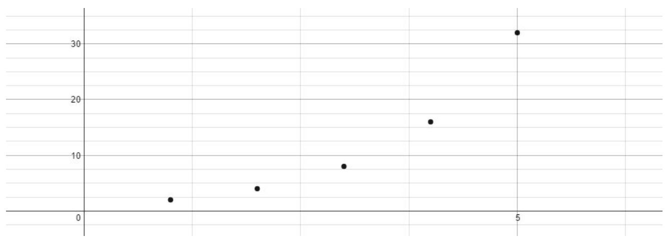


(d) $(2^n)_{n=1}^{\infty}$. Limit DNE.

1D number line



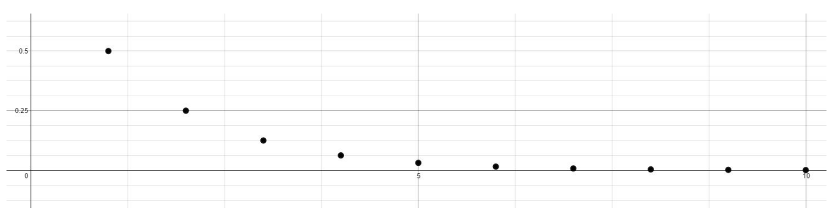
2D graph



(e) $\left(\frac{1}{2^n}\right)_{n=1}^{\infty}$. Limit $= 0$.

1D number line

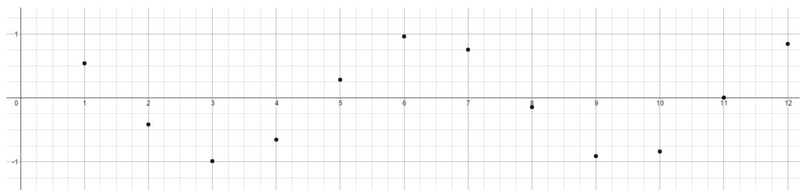
2D graph



(f) $(\cos(n))_{n=1}^{\infty}$. Limit DNE.

1D number line

2D graph



Proof: Does not converge
FSOC, suppose $\cos n$ does converge to some L. Because the range of $\cos n$ is $[-1, 1]$, $-1 \le L \le 1$. Since L is the limit point, consider all neighborhoods of L of the form $(L - \epsilon, L + \epsilon)$. Because $\cos n$ repeatedly cycles through values of its range from -1 to 1, whenever $|\epsilon| < 1$, there are infinitely many sequence terms outside of the neighborhood of L. Thus by No. 70, the sequence does not converge.

■

54. Developing Limit Intuition

Consider the sequence $(a_n)_{n=1}^{\infty} = (1.3 + \frac{1}{n})_{n=1}^{\infty}$.

(a) What is the limit L of this sequence?
$L = 1.3$

(b) If $\epsilon = 0.1$ then what can N be so that $|a_n - L| < \epsilon$ for all $n \geq N$?

$$|a_n - L| < \epsilon$$
$$|(1.3 - \frac{1}{N}) - 1.3| < 0.1$$
$$\frac{1}{N} < \frac{1}{10}$$
$$N > 10$$

(c) If $\epsilon = 0.05$ then what can N be so that $|a_n - L| < \epsilon$ for all $n \geq N$?

$$\frac{1}{N} < \frac{1}{20}$$
$$N > 20$$

(d) If $\epsilon = 10^{-4}$ then what can N be so that $|a_n - L| < \epsilon$ for all $n \geq N$?

$$\frac{1}{N} < \frac{1}{1000}$$
$$N > 1000$$

55. Convergence of $(1.3 + \frac{1}{n})_{n=1}^{\infty}$

Prove $(1.3 + \frac{1}{n})_{n=1}^{\infty}$ converges.
Let $\epsilon > 0$. Let $L = 1.3$. Let $N = \lceil \frac{1}{\epsilon} \rceil$. Let $n > N$.

$$|1.3 + \frac{1}{N} - 1.3| = |\frac{1}{N}|$$
$$\frac{1}{N} = \frac{1}{\lceil \frac{1}{\epsilon} \rceil} \leq \frac{1}{\frac{1}{\epsilon}} = \epsilon$$

Accordingly, $\forall\ \epsilon > 0$, $\frac{1}{N} \leq \epsilon\ \forall\ n > N$.

$\therefore$ $(1.3 + \frac{1}{n})_{n=1}^{\infty}$ converges.

■

56. Convergence of $(-10 + \frac{n}{n+1})_{n=1}^{\infty}$

Prove $(-10 + \frac{n}{n+1})_{n=1}^{\infty}$ converges.

Let $\epsilon > 0$. Let $L = -9$. Let $N = \lceil \frac{1-\epsilon}{\epsilon} \rceil$. Let $n > N$.

$$\begin{aligned} \left| -10 + \frac{N}{N+1} + 9 \right| &= \left| \frac{N}{N+1} - 1 \right| \\ &= \left| \frac{\frac{1}{\epsilon} - 1}{\frac{1}{\epsilon} - 1 + 1} - 1 \right| \\ &= \left| \epsilon \left(\frac{1}{\epsilon} - 1 \right) - 1 \right| \\ &= |1 - \epsilon - 1| \\ &= \epsilon \end{aligned}$$

Accordingly, $\forall\ \epsilon > 0$, $\left| \frac{N}{N+1} - 1 \right| \ \leq \epsilon\ \forall\ n > N$.

$\therefore\ (-10 + \frac{n}{n+1})_{n=1}^{\infty}$ converges.

■

57. Convergence of $(3 + \frac{1}{n^2})_{n=1}^{\infty}$

Prove $(3 + \frac{1}{n^2})_{n=1}^{\infty}$ converges.
Let $\epsilon > 0$. Let $L = 3$. Let $N = \left\lceil \sqrt{\frac{1}{\epsilon}} \right\rceil$. Let $n > N$.

$$|3 + \frac{1}{N^2} - 3| = |\frac{1}{N^2}|$$
$$\frac{1}{N^2} = \frac{1}{\left\lceil \sqrt{\frac{1}{\epsilon}} \right\rceil^2} \ \leq\ \frac{1}{\left(\sqrt{\frac{1}{\epsilon}} \right)^2} = \epsilon$$

Accordingly, $\forall\ \epsilon > 0$, $\frac{1}{N^2}\ \leq \epsilon\ \forall\ n > N$.

$\therefore\ (3 + \frac{1}{n^2})_{n=1}^{\infty}$ converges.

■

58. Convergence of a Constant

Prove that $(1.7)_{n=1}^{\infty}$ converges.

Let $L = 1.7$ and $N = 1$. Then for $\epsilon > 0$: $|1.7 - 1.7| = 0 < \epsilon$.

Accordingly, $\forall\ \epsilon > 0\ \exists\ N$ such that $|a_n - L| < \epsilon,\ \ \forall\ n \geq N$.

$\therefore$ $(1.7)_{n=1}^{\infty}$ converges.

■

59. Convergence of $((\frac{1}{2})^n)_{n=1}^{\infty}$

Prove that $((\frac{1}{2})^n)_{n=1}^{\infty}$ converges.

Let $\epsilon > 0$. Let $L = 0$. Let $N = \lceil \log_2(\frac{1}{\epsilon}) \rceil$. Let $n > N$.

$$\begin{aligned}
\left|\left(\frac{1}{2}\right)^n - 0\right| &= \left|\left(\frac{1}{2}\right)^n\right| \\
&= \left(\frac{1}{2}\right)^n \\
&= \frac{1}{2^n} \\
&< \frac{1}{2^N} \\
&= \frac{1}{2^{log_2(\frac{1}{\epsilon})}} \\
&= \frac{1}{\frac{1}{\epsilon}} \\
&= \epsilon
\end{aligned}$$

Accordingly, $\forall\ \epsilon > 0\ \exists\ N$ such that $\left|\left(\frac{1}{2}\right)^n\right| < \epsilon,\ \ \forall\ n \geq N$.

$\therefore$ $((\frac{1}{2})^n)_{n=1}^{\infty}$ converges.

■

60. Convergence of $((-1)^n)_{n=1}^{\infty}$

Prove that $((-1)^n)_{n=1}^{\infty}$ does not converge.

FSOC suppose $(-1)^n$ converges to $L \in \mathbb{R}$. Then:
$\forall\ \epsilon > 0,\ \ \exists N \in \mathbb{N} : |(-1)^n - L| < \epsilon\ \ \forall\ n > N$

Let $\epsilon = 1$. Then $\exists\ N \in \mathbb{N} : |(-1)^n - L| < 1\ \forall\ n > N$.

Case 1: $n > N$ and n is odd:

$$\begin{aligned} |(-1)^n - L| &< 1 \\ |-1 - L| &< 1 \\ -1 < -1 - L &< 1 \\ 0 < -L &< 2 \\ -2 < L &< 0 \end{aligned}$$

Case 2: $n > N$ and n is even:

$$\begin{aligned} |(-1)^n - L| &< 1 \\ |1 - L| &< 1 \\ -1 < 1 - L &< 1 \\ -2 < -L &< 0 \\ 2 > L &> 0 \end{aligned}$$

This is a contradiction because L can't satisfy both inequalities. Thus in the definition of convergence, $|a_n - L| < \epsilon$, no such L exists.

$\therefore$ $((-1)^n)_{n=1}^{\infty}$ does not converge.

$\blacksquare$

61. Sequence converging to α

Let $\alpha = \sup M$. Prove there exists a sequence $(a_n)_{n=1}^{\infty}$ that converges to α such that $a_n \in M$ for all natural numbers n. (A similar statement is also true for $\inf M$.)

Case 1: $\alpha \in M$. Then let $a_n = \alpha$ and $L = \alpha$, so that a_n converges to L by No. 58 (convergence of a constant).

Case 2: $\alpha \notin M$. Then by the definition of supremum, $\forall\ m \in M,\ \exists\ m' \in M$ such that $m < m' < \alpha$.Let $a_1 = m'$. Likewise, for $m'' \in M$ such that $m' < m'' < \alpha$. Let $a_2 = m''$. Continue in this manner to establish $a_n \in M$ for all natural numbers n. Furthermore, because $a_n > ... > a_3 > a_2 > a_1$, the sequence is monotone increasing and bounded above by its least upper bound, the supremum of $M = \alpha$. Thus by No. 62, a_n converges to α.

$\therefore$ For both $\alpha \in M$ and $\alpha \notin M$, there exists a sequence $(a_n)_{n=1}^{\infty}$ that converges to α such that $a_n \in M$ for all natural numbers n.

$\blacksquare$

62. Convergence of Monotonically Increasing Above Bounded Sequence

Suppose a sequence $(a_n)_{n=1}^{\infty}$ is monotonically increasing. That is, $a_n \leq a_{n+1}\ \forall\ n = 1, 2, 3, ...$ Assume $(a_n)_{n=1}^{\infty}$ is also bounded above. Prove $(a_n)_{n=1}^{\infty}$ converges.

Because $(a_n)_{n=1}^{\infty}$ is bounded above, by the Least Upper Bound Property, the supremum of $(a_n)_{n=1}^{\infty}$ exists. Let $L = \sup a$.

Because $(a_n)_{n=1}^{\infty}$ is monotonically increasing, $a_n \leq a_{n+1}$ so a_{n+1} is closer to or the same distance from L than a_n, or $|a_{n+1} - L| \leq |a_n - L|$. Thus, there must exist some N at which the difference between a_N and $\sup a$ must only decrease as n increases past N.

Let $\epsilon > 0$. Let $L = \sup a$. At N, let $a_N = L - \epsilon$. Let $n > N$. Then $|a_N - L| = |\sup a - \epsilon - \sup a| = \epsilon \leq \epsilon$. Thus the N that satisfies $a_N = \sup A - \epsilon$ is the N at which convergence begins. Accordingly,

$$\forall\, \epsilon > 0,\ \exists\ N : |a_n - L| < \epsilon,\ \forall\ n \geq N$$

$\therefore$ When monotonically increasing and bounded above, $(a_n)_{n=1}^{\infty}$ converges.

$\blacksquare$

7 Properties of Convergent Sequences

64. Convergent sequence ⇒ Bounded point set

Prove that if a sequence $(a_n)_{n=1}^{\infty}$ converges to L, then the point set $M = \{a_n : n \in \mathbb{N}\}$ is bounded.

$$(a_n) \to L \;\Rightarrow\; \forall\, \epsilon > 0,\; \exists\; N \in \mathbb{N} :$$

$$\begin{aligned} |a_n - L| &< \epsilon \;\; \forall\;\; n \geq N \\ ||a_n| - |L|| < |a_n - L| &< \epsilon \\ |a_n| - |L| &< \epsilon \\ |a_n| &< \epsilon + |L| \end{aligned}$$

Let $C = \epsilon + |L|$. Thus $|a_n| \leq C$ for all $n > N$. For the finite number of elements of a_n such that $1 \leq n < N$, let $X = \max(|a_n - L|)$. Because X is the greatest distance between a_n and L for n in the interval $1 \leq n < N$, $|a_n - L| < X$. Thus:

$$\begin{aligned} |a_n - L| &< X \;\; \forall\;\; 1 \leq n > N \\ ||a_n| - |L|| < |a_n - L| &< X \\ |a_n| - |L| &< X \\ |a_n| &< X + |L| \end{aligned}$$

Let $D = X + |L|$. Thus $|a_n| < D$ for all $1 \leq n < N$. Accordingly, for $n \geq N$, (a_n) is bounded and for $1 \leq n < N$, (a_n) is bounded. So for all $n \in \mathbb{N}$, a_n is bounded.

$\therefore$ $(a_n)_{n=1}^{\infty}$ converges to $L \Rightarrow M = \{a_n : n \in \mathbb{N}\}$ is bounded.

■

65. $(ca_n)_{n=1}^{\infty}$ converges to cL

Suppose $(a_n)_{n=1}^{\infty}$ converges to L and c is a constant. Prove $(ca_n)_{n=1}^{\infty}$ converges to cL.

Because $(a_n)_{n=1}^{\infty}$ converges to L, for all $\epsilon > 0$, there exists $N > 0$ such that $|a_n - L| < \epsilon,\ \forall\, n \geq N$. Because this is true *for all* $\epsilon > 0$, it is true for $\frac{\epsilon}{c}$:

$$\begin{aligned} |a_n - L| &< \frac{\epsilon}{c} \\ -\frac{\epsilon}{c} < a_n - L &< \frac{\epsilon}{c} \\ c\left(-\frac{\epsilon}{c}\right) < ca_n - cL &< c\left(\frac{\epsilon}{c}\right) \\ -\epsilon < can - cL &< \epsilon \\ |can - cL| &< \epsilon \end{aligned}$$

Accordingly, because $\forall\, \epsilon > 0,\ \exists\, N\ :\ |a_n - L| < \epsilon,\ \ \forall\, n > N$, by the same N $\forall\, \epsilon > 0,\ \ |ca_n - cL| < \epsilon,\ \ \forall\, n > N$.

$\therefore (ca_n)_{n=1}^{\infty}$ converges to cL.

■

66. Convergence of Added Converging Functions

Suppose $(a_n)_{n=1}^{\infty}$ converges to L and $(b_n)_{n=1}^{\infty}$ converges to K. Prove that $(a_n + b_n)_{n=1}^{\infty}$ converges to $L + K$.

$$\begin{aligned} (a_n) \to L \ \Rightarrow\ |a_n - L| &< \epsilon\ \forall\ n > N_a \\ |a_n - L| &< \frac{\epsilon}{2}\ \forall\ n > N_a \end{aligned}$$

$$\begin{aligned} (b_n) \to L \ \Rightarrow\ |b_n - K| &< \epsilon\ \forall\ n > N_b \\ |b_n - K| &< \frac{\epsilon}{2}\ \forall\ n > N_b \end{aligned}$$

Let $N = \max(N_a, N_b)$. Then:

$$\begin{aligned} |a_n - L| + |b_n - K| &< \frac{\epsilon}{2} + \frac{\epsilon}{2}\ \forall\ n > N \\ |(a_n - L) + (b_n - K)| \leq |a_n - L| + |b_n - K| &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ \forall\ n > N \\ |(a_n + b_n) - (L + K)| < \epsilon\ \forall\ n > N \end{aligned}$$

Thus by the definition of convergence, $(a_n + b_n)$ converges to $L + K$.
$\therefore (a_n)_{n=1}^{\infty} \to L$ and $(b_n)_{n=1}^{\infty} \to K \Rightarrow (a_n + b_n) \to L + K$.

■

67. Convergence of Multiplied Converging Functions

Suppose $(a_n)_{n=1}^{\infty}$ converges to L and $(b_n)_{n=1}^{\infty}$ converges to K. Prove that $(a_nb_n)_{n=1}^{\infty}$ converges to LK.

$$\begin{aligned}|a_nb_n - LK| &= |a_nb_n + Lb_n - Lb_n - LK| \\ &= |b_n(a_n - L) + L(b_n - K)| \\ &\leq |b_n(a_n - L)| + |L(b_n - K)| \\ &= |b_n||a_n - L| + |L(b_nK)|\end{aligned}$$

By No. 64 because $b_n \to K$, b_n is bounded. Thus, for some constant C, $|b_n| < C \ \forall \ n \in \mathbb{N}$.

$$\begin{aligned}&\leq C|a_n - L| + |L(b_n - K)| \\ &\leq |Ca_n - CL| + |Lb_n - LK|\end{aligned}$$

By No. 65, Ca_n converges to CL and Lb_n converges to LK. By No. 66, the sum of these two converging sequences converges to the sum of their limits, $CL + LK$. Thus $\forall \ \epsilon > 0$:

$$|Ca_n + Lb_n - (CL + LK)| \leq |Ca_n - CL| + |Lb_n - LK| < \epsilon$$

By the definition of convergence:

$$\forall \ \epsilon > 0, \ \exists \ N : |Ca_n + Lb_n - (CL + LK)| < \epsilon, \ \forall \ n > N$$

Because we have shown the following inquality:

$$|a_nb_n - LK| \ \leq \ |Ca_n + Lb_n - (CL + LK)| < \epsilon$$

we can apply the definition of convergence as follows:

$$\forall \ \epsilon > 0, \ \exists \ N : |a_nb_n - LK| < \epsilon, \ \forall \ n > N.$$

$\therefore$ $(a_nb_n)_{n=1}^{\infty}$ converges to LK.

■

68. Convergence of Divided Converging Functions

Suppose $(a_n)_{n=1}^{\infty}$ converges to L and $(b_n)_{n=1}^{\infty}$ converges to K. Prove that $(\frac{a_n}{b_n})_{n=1}^{\infty}$ converges to $\frac{L}{K}$.

Because $(b_n)_{n=1}^{\infty} \to K \neq 0$, by No. 70, all except finitely many elements of b_n are in any neighborhood of K, the interval $(K-\epsilon, K+\epsilon)$. Because there are only finitely elements outside of $(K-\epsilon, K+\epsilon)$, let these outside elements be Q, and there exists a minimum element $q \in Q$ outside of the neighborhood. Let $M = \min(|q|, |K-\epsilon|, |K-\epsilon|)$. Then $M < |b_n|$.

Because $(b_n)_{n=1}^{\infty}$ converges to K,

$$\forall\ \epsilon > 0,\ \exists\ N\ \in \mathbb{N} : |b_n - K| < \epsilon,\ \forall\ n > N$$

Because this is true for all epsilon, it is true for $KM\epsilon$:

$$\begin{aligned}
|b_n - K| &< KM\epsilon \\
\frac{1}{KM}|b_n - K| &< \frac{1}{KM}(KM)\epsilon \\
\frac{1}{KM}|b_n - K| &< \epsilon \\
\frac{1}{K|b_n|}|b_n - K| \leq \frac{1}{KM}|b_n - K| &< \epsilon \\
\left|\frac{b_n - K}{Kb_n}\right| &< \epsilon \\
\left|\frac{1}{b_n} - \frac{1}{K}\right| &< \epsilon
\end{aligned}$$

Accordingly, $\forall\ \epsilon > 0,\ \exists\ N\ \in \mathbb{N} : \left|\frac{1}{b_n} - \frac{1}{K}\right| < \epsilon,\ \ \forall\ n > N$, so by the definition of convergence, $(\frac{1}{b_n})_{n=1}^{\infty}$ converges to $\frac{1}{K}$.

By 67, because $(a_n)_{n=1}^{\infty}$ converges to L and $(\frac{1}{b_n})_{n=1}^{\infty}$ converges to $\frac{1}{K}$:

$$\left(a_n\left(\frac{1}{b_n}\right)\right)_{n=1}^{\infty} = (L)\left(\frac{1}{K}\right) = \frac{L}{K}$$

$\therefore\ (\frac{a_n}{b_n})_{n=1}^{\infty}$ converges to $\frac{L}{K}$.

■

70. $a_n \to L \iff$ neighborhoods of L contain all but finitely many of the terms of $(a_n)_{n=1}^{\infty}$

Let $(a_n)_{n=1}^{\infty}$ be a sequence. Show that $a_n \to L$ if and only if every neighborhood of L contains all but finitely many of the terms of $(a_n)_{n=1}^{\infty}$.

1. First assume $a_n \to L$. Then by the definition of convergence, $\forall\, \epsilon > 0$, there exists $N > 0$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. Accordingly

   $$-\epsilon + L < a_n < \epsilon + L \;\forall\; n > N$$

   So there is some N, past which all of the sequence is within every neighborhood of L. The elements of the sequence such that $1 \leq n < N$ are outside of the neighborhood of L but are finitely numbered. Accordingly, if $a_n \to L$, then every neighborhood of L contains all except finitely many of the terms of $(a_n)_{n=1}^{\infty}$.

2. Then assume every neighborhood of L contains all but finitely many of the terms of $(a_n)_{n=1}^{\infty}$. For $\epsilon > 0$, let all neighborhoods of L be $(L - \epsilon, L + \epsilon)$ so that $L - \epsilon < a_n < L + \epsilon$ for all $n > N$ but not $1 \leq n < N$. Thus the elements of the sequence when $1 \leq n < N$ are finitely numbered, and the elements of the sequence $n > N$ comprise the rest of the sequence. Then by the definition of absolute value, for all $\epsilon > 0$, there exists N such that $|a_n - L| < \epsilon$ for all $n > N$, so a_n converges to L. Accordingly, if every neighborhood of L contains all except finitely many of the terms of $(a_n)_{n=1}^{\infty}$, then $a_n \to L$.

■

71. Uniqueness of Limits

Let $(a_n)_{n=1}^{\infty}$ be a sequence. Let L and L' be in $\mathbb{R}$. Show that if $(a_n)_{n=1}^{\infty}$ converges to L and L', then $L = L'$.

FSOC, suppose $L \neq L'$. Then $\exists\, d \in \mathbb{R}$ such that $d = |L - L'|$.

By the definition of convergence, $\forall\, \epsilon > 0$, $\exists\, N$:

$$|a_n - L| < \epsilon \text{ and } |a_n - L'| < \epsilon$$

Then let $\epsilon = \frac{d}{2}$:

$$|a_n - L| < \frac{d}{2} \text{ and } |a_n - L'| < \frac{d}{2}$$

Accordingly:

$$d = |L - L'| = |L + a_n - a_n - L'| \leq |L - a_n| + |a_n - L|$$
$$= |a_n - L| + |a_n - L| < \frac{d}{2} + \frac{d}{2}$$

This implies that $d < d$, which is a contradiction.

$\therefore$ If $(a_n)_{n=1}^{\infty} \to L$ and $(a_n)_{n=1}^{\infty} \to L'$, then L = L'.

■

72. L = Limit point of $S \Rightarrow$ convergence to L

If $S \subset \mathbb{R}$ and L is a limit point of S then there is a sequence $(a_n)_{n=1}^{\infty}$ in S such that $L = \lim_{n\to\infty} a_n$.

Because L is a limit point of S, every open interval of with L contains a point of S different from L.

$$\forall\, \epsilon > 0,\ \exists\ q \in (L - \epsilon, L + \epsilon), q \in S, q \neq L$$

So for some ϵ_1, $\exists\, q_1$, call it a_1.
Let $\epsilon_2 = \frac{|a_1 - L|}{2}$. Thus $\exists\, q_2 \in S$, $q_2 \neq q_1$ call it a_2.
Likewise, let $\epsilon_3 = \frac{|a_2 - L|}{2}$. Thus $\exists\, q_3 \in S$, $q_3 \neq q_2$ call it a_3.
Continuing in this manner of choosing $\epsilon_n = \frac{|a_{n-1} - L|}{2}$, by the definition of a limit point, for all ϵ there must always exist a unique a_n. Because $\epsilon < a_1 < L$, the first term of the sequence is within the neighborhood of L. Every term after a_1 is greater than a_1, so a_n is always within the closed interval $(L - \epsilon, L + \epsilon)$ $\forall\, n > 1$. Accordingly, $\forall\, \epsilon > 0\ \exists\, N, N = 1\ :\ |a_n - L| < \epsilon,\ \forall\ n > N$, so $(a_n)_{n=1}^{\infty}$ converges to L.

$\therefore$ There is a sequence $(a_n)_{n=1}^{\infty}$ in S such that $L = \lim_{n\to\infty} a_n$.

■

8 Continuity

80. Continuity of Linear function

Let $f(x) = 2x$ for all $x \in \mathbb{R}$. Prove that f is continuous at $c = 3$.

(a) The domain of $f(x)$ is all $x \in \mathbb{R}$, so $c = 3$ is in the domain of $f(x)$.

(b) Let $a_n \to c = 3$. Let $f(a_n) = 2(a_n)$. Let $b_n = 2$ such that $b_n \to 2$ by No. 58. Thus $f(a_n) = (b_n)(a_n)$ which converges to $(2)(3) = 6 = f(3)$ by No. 67. Accordingly, while $a_n \to c$, $f(a_n) \to f(c)$.

$\therefore f$ is continuous at $c = 3$.

■

81. Continuity of Quadratic Function

Let $f(x) = x^2$ for all $x \in \mathbb{R}$. Prove that f is continuous at $c = -2$.

(a) The domain of f is all $x \in \mathbb{R}$. Because $c = -2 \in \mathbb{R}$, $c \in \text{domain}(f)$.

(b) Let $a_n \to -2$.
Notice $f(a_n) = (a_n)^2 = (a_n)(a_n)$. Because $a_n \to -2$, by No. 67:

$$f(a_n) = (a_n)(a_n) \to (-2)(-2) = 4 = (-2)^2 = f(-2)$$

Thus while $a_n \to -2$, $f(a_n) \to f(-2)$.

$\therefore f$ is continuous at $c = -2$.

■

82. Continuity of Constant Function

Let $f(x) = \pi + 1$ for all $x \in \mathbb{R}$. Prove that f is continuous for all real numbers x.

The domain of f is $\mathbb{R}$, for some c in $\mathbb{R}$, c is in the domain of f.

Note: $\forall\ x \in \mathbb{R},\ f(x) = \pi + 1$. Thus $\forall\ x \in \mathbb{R}$:

$$\forall\ \epsilon > 0,\ |f(a_n) - f(c)| = |(\pi + 1) - (\pi + 1)| = |0| < \epsilon$$

Accordingly, $|f(a_n) - f(c)|$ is always less than ϵ, and by No. 58 the constant term $|f(a_n) - f(c)|$ is convergent for any c, so it is always true that $f(a_n) \to f(c)$. Let $a_n \to c$. Accordingly, when $a_n \to c$, $f(a_n) \to f(c)$.

$\therefore f$ is continuous for all real numbers x.

■

83. Interval with $f(x) > 0$ for continuous f

If a function f is continuous on $[a, b]$ and there exists $x \in (a, b)$ such that $f(x) > 0$, then there exists an open interval T, containing x, such that $f(t) > 0$ for all $t \in T$.

Because f is continuous on $[a, b]$:

$$\forall\ \epsilon > 0,\ \exists\ \delta > 0\ :\ |f(y) - f(x)| < \epsilon,\ \forall\ \ |y - x| < \delta$$

Because $f(x) > 0$, let $\epsilon = f(x)$. Accordingly:

$$\begin{aligned} |f(y) - f(x)| < f(x) &\quad \text{is true } \forall\ y \text{ such that} \quad |y - x| < \delta \\ -f(x) < f(y) - f(x) < f(x) &\quad \text{is true } \forall\ y \text{ such that} \ -\delta < y - x < \delta \\ 0 < f(y) < 2f(x) &\quad \text{is true } \forall\ y \text{ such that} \ -\delta + x < y < \delta + x \\ 0 < f(y) < 2f(x) &\quad \text{is true } \forall\ y \text{ such that} \ -\delta + x < y < \delta + x \end{aligned}$$

Let $T = (x - \delta,\ x + \delta)$ and $t = y$ so that $\forall\ t \in T,\ \ f(t) > 0$.

$\therefore\quad \exists\ T$ with $x \in T$ such that $f(t) > 0,\ \ \forall\ t \in T$.

■

84. Continuity of $f(x) = x^n$ for all real numbers

Let $f(x) = x^n$ for some $n \in \mathbb{N}$. Prove that f is continuous for all $x \in \mathbb{R}$.

(a) The domain of f is all $x \in \mathbb{R}$. Thus all $x \in \mathbb{R}$ is in the domain of f.

(b) For $x \in \mathbb{R}$, let $a_k \to x$.
Notice $f(a_k) = (a_k)(a_k + 1)...(a_n)$. Because $a_k \to x$, by No. 67:

$$f(a_k) = (a_k)(a_k + 1)...(a_n) \to (x)(x)...(x) = x^n = f(x)$$

Thus while $a_k \to x$, $f(a_k) \to f(x)$.

$\therefore f$ is continuous for all $x \in \mathbb{R}$.

■

86. Equivalence of Continuity Definitions 1 and 2

Prove that the two definitions of continuity are equivalent.

($\Rightarrow$)First assume Definition No. 2 and the the condition of Definition of No. 1:

Definition No. 2:

$$\forall\ \epsilon > 0,\ \exists\ \delta > 0,\ \forall\, y : |y - x| < \delta\ \Rightarrow\ |f(y) - f(x)| < \epsilon$$

Condition of Definition No. 1:

$$a_n\ \rightarrow\ c\ \Rightarrow\ \forall\ \delta > 0,\ \exists\ N\ : |a_n - c| < \delta\ \forall\, n > N$$

Let $y = a_n$ and $c = x$ so that the condition of Definition No. 1 becomes:

$$\exists\ N\ :\ |y - x| < \delta,\ \forall\, n > N$$

Then by the Definition of No. 2:

$$\forall\, \epsilon > 0, \exists\ N\ :\ |f(a_n) - f(c)| < \epsilon,\ \forall\, n > N$$

Accordingly, for every $a_n\ \rightarrow c$, $f(a_n)\ \rightarrow c$.

($\Leftarrow$)Then assume Definition No. 1 and the condition of Definition NO. 2:

Definition No. 1:

$$\forall\ a_n \rightarrow c, f(a_n) \rightarrow f(c)$$

Condition of Definition No. 2:

$$\forall\ \epsilon > 0,\ \exists\ \delta > 0, \exists\, y : |y - x| < \delta$$

FSOC, suppose that while $|y - x| < \delta$, $|f(y) - f(x)| \geq \epsilon$. So while $\delta = \frac{1}{n}$, $\exists\ y\ :\ |f(y) - f(x)| \geq \epsilon$. If $y = a_n$ and $x = c$, then $|f(a_n) - f(c)| \geq \epsilon$. But this implies that $f(a_n)$ does not converge to $f(c)$, which contradicts Definition No. 1. Accordingly, $|f(a_n) - f(c)|$ must be $< \epsilon$, and thus $|f(y) - f(x)| < \epsilon$.

$\therefore$ The first two definitions of continuity are equivalent.

■

87. Continuous + Continuous = Continuous

Let f and g be continuous functions on $[a,b]$. Prove the functions $h = f + g$ is also continuous on $[a,b]$.

Because f is continuous on $[a,b]$:

$$\forall\ \epsilon > 0,\ \exists\ \delta_f : |f(y) - f(x)| < \frac{\epsilon}{2},\ \ \forall\ y : |y - x| < \delta_f$$

Similarly, because g is continuous on $[a,b]$:

$$\forall\ \epsilon > 0,\ \exists\ \delta_g : |g(y) - g(x)| < \frac{\epsilon}{2},\ \ \forall\ y : |y - x| < \delta_g$$

Both of these statements are simultaneously true for $\delta_h = \min(\delta_f, \delta_g)$.

$$\begin{aligned}
|h(y) - h(x)| &= |(f(x) + f(y)) - (g(y) + g(x))| \\
&= |f(x) + f(y) - g(y) - g(x)| \\
&= |(f(y) - g(y)) + (f(x) - g(x))| \\
&\leq |f(y) - g(y)| + |f(x) - g(x)| \\
&< \left|\frac{\epsilon}{2}\right| + \left|\frac{\epsilon}{2}\right| \\
&< \left|\frac{\epsilon}{2} + \frac{\epsilon}{2}\right| \\
&< \epsilon
\end{aligned}$$

Accordingly, with h representing the sum of f and g:

$$\forall\ \epsilon > 0,\ \exists\ \delta_h : |h(y) - h(x)| < \epsilon,\ \ \forall\ y : |y - x| < \delta_h$$

$\therefore$ For two continuous functions f and g on $[a,b]$, $h = f + g$ is also continuous on $[a,b]$.

$\blacksquare$

88. (Continuous)(Continuous) = Continuous

Let f and g be continuous functions on $[a, b]$. Prove the functions $h = f \cdot g$ is also continuous on $[a, b]$.

Because f is continuous, for all $x \in [a, b]$, while $a_n \to x$, $f(a_n) \to f(x)$.

Because g is continuous, for all $y \in [a, b]$, while $b_n \to x$, $g(b_n) \to g(x)$.

By No. 67, $(a_n)(b_n) \to (x)(x)$.

By No. 67, $f(a_n)(b_n) \to f(x)g(x) = h(x)$.

Accordingly, for every $(a_n)(b_n) \to (x)(x)$, $f((a_n)(b_n)) \to f(x)g(x) = h((x)(x))$.

$\therefore$ $h(x) = f(x)g(x)$ is continuous on $[a, b]$.

■

89. Discontinuity Example

Prove that the function f is not continuous at $x = c$ if $f(x) = 1$ for all $x \neq c$ and $f(c) = 2$.

Let $a_n \to c$. Then by No. 51, $\forall\ \delta > 0$, $|a_n - c| < \delta$. Consider $\epsilon = \frac{1}{2}$. Then:

$$|f(x) - f(c)| = |1 - 2| = 1 > \epsilon = \frac{1}{2}$$

Because when $|a_n - c| < \delta$, $\exists\ \epsilon : |f(x) - f(c)| > \epsilon$, by No. 85, $f(x)$ does not converge to $f(c)$ while a_n converges to c.

$\therefore$ f is not continuous at $x = c$.

■

90. Discontinuity Example

Prove that for any real number x,

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not continuous.

FSOC suppose f is continuous for all $x \in \mathbb{R}$. Then while $(a_n) \to c$, WLOG for $c \in \mathbb{Q}$, $f(a_n) \to f(c) = 1$. Consider $a_n \notin \mathbb{Q}$. Then $f(a_n) \to 1$ but this is a contradiction because $f(a_n) = 0$. Thus while $a_n \to c$, $f(a_n)$ does not converge to $f(c)$.

$\therefore f$ is not continuous for any real number.

■

92. Equivalence of Continuity Definitions

Prove that the third definition of continuity is equivalent to the other two.

Definition 3:
A function f is said to be continuous at a point x if for every $n \in \mathbb{N}$ there exists an $m \in \mathbb{N}$ such that $|f(y) - f(x)| < \frac{1}{n}$ for all y such that $|y - x| < \frac{1}{m}$.

Definition 2:
A function f is said to be continuous at a point x if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all y such that $|y - x| < \delta$.

(2 ⇒ 3) Assume the condition of Definition 3 and Definition 2, $\forall \epsilon, \delta > 0$:

$$|y - x| < \frac{1}{m} \text{ and } |y - x| < \delta \Rightarrow \exists\ \delta : |f(y) - f(x)| < \epsilon$$

Consider $0 < \epsilon < \frac{1}{n}$ and $0 < \delta < \frac{1}{m}$. Accordingly,

$$|y - x| < \delta < \frac{1}{m}$$

$$|f(y) - f(x)| < \epsilon < \frac{1}{n}$$

Thus whenever $|y - x| < \frac{1}{m}$, it is true that $|f(y) - f(x)| < \frac{1}{n}$.

$$\forall\ n \in \mathbb{N},\ \ \exists\ \ m \in \mathbb{N} : |f(y) - f(x)| < \frac{1}{n}\ \ \forall\ y : |y - x| < \frac{1}{m}$$

(3 ⇒ 2) Assume the condition of Definition 2 and Definition 3, $\forall \epsilon, \delta > 0$:

$$|y - x| < \delta \text{ and } |y - x| < \frac{1}{m}\ \Rightarrow \exists\ \in \mathbb{N} : |f(y) - f(x)| < \frac{1}{n}$$

Consider $0 < \frac{1}{n}\epsilon$ and $0 < \frac{1}{m}\delta$. Accordingly,

$$|y - x| < \frac{1}{m}\delta$$

$$|f(y) - f(x)| < \frac{1}{n} < \epsilon$$

Thus whenever $|y - x| < \delta$, it is true that $|f(y) - f(x)| < \epsilon$.

$$\forall\ \epsilon > 0\ \exists\ \ \delta > 0 : |f(y) - f(x)| < epsilon\ \ \forall\ y : |y - x| < \delta$$

Accordingly, Definitions 2 and 3 are equivalent. By 86, Definitions 1 and 2 are equivalent.

∴ Definitions 1, 2, and 3 are equivalent.

■

93. Continuous at $\mathbb{Q}$

Suppose f has domain $(0,1)$ and is defined to be

$$f(x) = \begin{cases} \frac{1}{n} & x = \frac{m}{n} \text{ in lowest terms} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Prove that f is not continuous at every rational number in $(0,1)$ and that f is continuous at every irrational in $(0,1)$.

For $x \in \mathbb{Q}$: Consider $(a_n) = x + \frac{\Phi}{n}$ so that $(a_n) \to x$, and thus $f(x) \neq 0$. Because $(x + \frac{\Phi}{n}) \notin \mathbb{Q}$, $f(x + \frac{\Phi}{n}) = 0$ and thus $f(a_n) \to 0 \neq f(x)$. Accordingly, while $a_n \to x$, $f(a_n)$ does not converge to $f(x)$ and thus f is not continuous at the rationals.

For $r \notin \mathbb{Q}$: For all $\epsilon > 0$, there exists $\frac{1}{n} < \epsilon$. Consider a fixed N such that $\forall\, n \in \mathbb{N}$, $n > N$, and thus $\frac{1}{n} < \frac{1}{N} \Rightarrow \frac{m}{n} < \frac{1}{N}$. So $\frac{1}{N} < \epsilon$. Because there is a finite number of points for $f(r) > \epsilon$, there is a minimum distance d from irrational r to rational x such that $f(x) > \epsilon$. Using this minimum distance, there exists δ, $0 < \delta < d$.

In the interval $(0, \delta)$, there is no rational $\frac{1}{n_2} : n_2 < n$. Thus for all y such that $|y - x| < \delta$, either $0 < \frac{1}{N} < \epsilon$ or $|f(y) - f(x)| = 0 < \epsilon$. Either way, for all $n > N$, by Definition No. 2 of continuity, f is continuous at the irrationals.

$\therefore$ On the interval $(0,1)$ f is not continuous at the rationals and continuous at the irrationals.

■

94. Continuity of $f(x) = \frac{1}{x}$

Show that the function $f(x) = \frac{1}{x}$ is continuous over its domain.

Let $a_n \to x$ such that $\forall\, n \in \mathbb{R}$, $a_n \neq 0$. Let $b_n = 1$. By No. 58, $b_n \to 1$. Then using No. 68:

$$f(a_n) = \frac{1}{a_n} = \frac{b_n}{a_n} \to \frac{1}{x} = f(x)$$

Thus while $a_n \to x$, $f(a_n) \to f(x)$, so by No. 78 $f(x)$ is continuous over its domain.

■

9 More on Continuity and Convergence

95. Limits are less extreme than Bounds

Let L be a constant. Suppose $(a_n)_{n=1}^{\infty}$ satisfies the property that $a_n \leq L$, for all $n \in \mathbb{N}$. If $a_n \to \alpha$, prove that $\alpha \leq L$. State and prove an analogous theorem for $a_n \geq L$.

FSOC, suppose $\alpha > L$. Consider the interval $(L, L+\alpha)$. Because $a_n \leq L \ \forall\ n \in \mathbb{N}$, for all $n \in \mathbb{N}$ the interval $(L, L+\alpha)$, does not contain a_n.

Accordingly the neighborhood of α that is the interval $(L, L+\alpha)$ contains no terms of a_n, and so does not contain all except finitely many of a_n. Thus by No. 70, a_n can not converge to α. This is a contradiction because we assumed $a_n \to \alpha$.

$\therefore\ a_n \leq L$ and $a_n \to \alpha \Rightarrow \alpha \leq L$

Similarly, if $(a_n)_{n=1}^{\infty}$ satisfies the property that $a_n \geq L$, for all $n \in \mathbb{N}$ and $a_n \to \alpha$, then $\alpha \geq L$.

FSOC, suppose $\alpha < L$. Consider the interval $(L-\alpha, L)$. Because $a_n \geq L \ \forall\ n \in \mathbb{N}$, for all $n \in \mathbb{N}$ the interval $(L-\alpha, L)$, does not contain a_n.

Accordingly the neighborhood of α that is the interval $(L-\alpha, L)$ contains no terms of a_n, and so does not contain all except finitely many of a_n. Thus by No. 70, a_n can not converge to α. This is a contradiction because we assumed $a_n \to \alpha$.

$\therefore\ a_n \geq L$ and $a_n \to \alpha \Rightarrow \alpha \geq L$

■

97. $\exists\, x \in \mathbb{R} : \cos(x) = x$

Show there is some $x \in \mathbb{R}$ so that $\cos(x) = x$.

Consider the function $f(x)$ such that $f(x) = \cos(x) - x$. We know that $\cos(x)$ is a continuous function and $-x$ is a continuous function by No. 82, No. 80, and No. 88 (Reference Exam problem No. 3). By No. 87, the addition of two continuous functions is also continuous, so $f(x)$ is continuous. By No. 96, if we can find an a and b such that $f(a) < 0$ and $f(b) > 0$, then continuity of $f(x)$ implies that $f(x) = 0$ for some $x \in \mathbb{R}$. Choose $a = \pi$ so that $f(a) = (-1 - \pi) < 0$ and $b = 0$ so that $f(b) = (1 + 0) > 0$. Accordingly there exists a and a b such that $f(a) < 0$ and $f(b) > 0$, so there must exists some x such that $f(x) = 0 \Rightarrow \cos(x) - x = 0 \Rightarrow \cos(x) = x$.

■

99. Temperature

Let $T(x)$ represent the temperature on the surface of the Earth, which we assume is a sphere. Thus $T : \mathbb{R}^3 \to \mathbb{R}$, where the point is a point on the surface of the Earth and the output is a temperature. Show there exists a point c on the surface such that $T(c) = T(-c)$.

Let $f(x) = T(c) - T(-c)$ so that $f(-x) = T(-c) - T(c)$. If $f(x) \neq 0$, then either $f(x) > 0$ or $f(x) < 0$:

$$f(x) > 0 \Rightarrow T(x) - T(-x) > 0 \Rightarrow T(-x) - T(x) < 0 \Rightarrow f(-x) < 0$$

$$f(x) < 0 \Rightarrow T(x) - T(-x) < 0 \Rightarrow T(-x) - T(x) > 0 \Rightarrow f(x) > 0$$

Accordingly, $f(-x) < 0 < f(x))$. If temperature T is in fact continuous, then the addition of two temperature functions is also continuous by No. 87, and so f is continuous. If f is continuous, then by No. 96, there must exist c such that $f(c) = 0$ which would give $T(c) = T(-c)$. However, temperature is not continuous. Consider a cave.

■