1. Find a Cartesian equation for the curve below and identify it:

$$r^2 \sin 2\theta = 1$$

In order to convert polar equations (with variables r and θ) to Cartesian form (with variables x and y), we will need to use the following equations that relate polar and Cartesian coordinates:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Accordingly, in the given equation $r^2 \sin 2\theta = 1$ we want to look for some form of $x = r\cos\theta$ or $y = r\sin\theta$ so that we can substitute these for x and y. Because we are considering 2θ rather than just θ we will need to use some Trigonometric Identity. After consulting our Trig Identities looking for one that involves $\sin 2\theta$ we use the following:

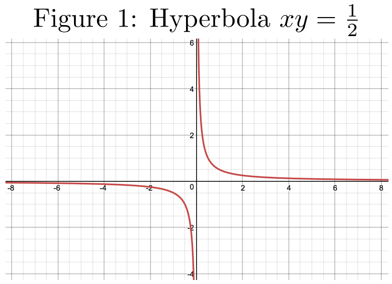
$$\sin 2\theta = 2\sin\theta\cos\theta$$

By substitution:

$$\begin{aligned} r^2 \sin 2\theta &= 1 \\ r^2(2\sin\theta\cos\theta) &= 1 \\ (r)(r)(2)(\sin\theta\cos\theta) &= 1 \\ (2)(r\sin\theta)(r\cos\theta) &= 1 \\ 2xy &= 1 \\ xy &= \frac{1}{2} \end{aligned}$$

We can then recognize $xy = \frac{1}{2}$ as a hyperbola, or rewrite it as $y = \frac{1}{2x}$ and graph to identify the following hyperbola:

Figure 1: Hyperbola $xy = \frac{1}{2}$

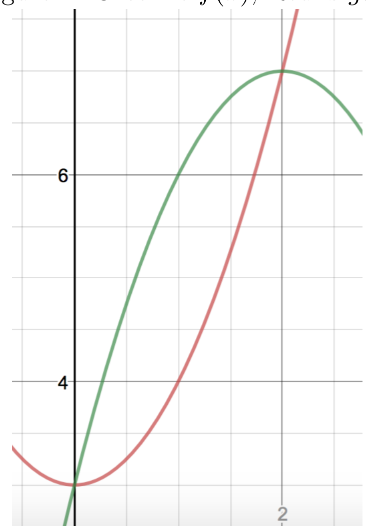


2. Find the centroid for an area defined by the equations:

$$y = x^2 + 3 \quad \text{and} \quad y = -(x-2)^2 + 7$$

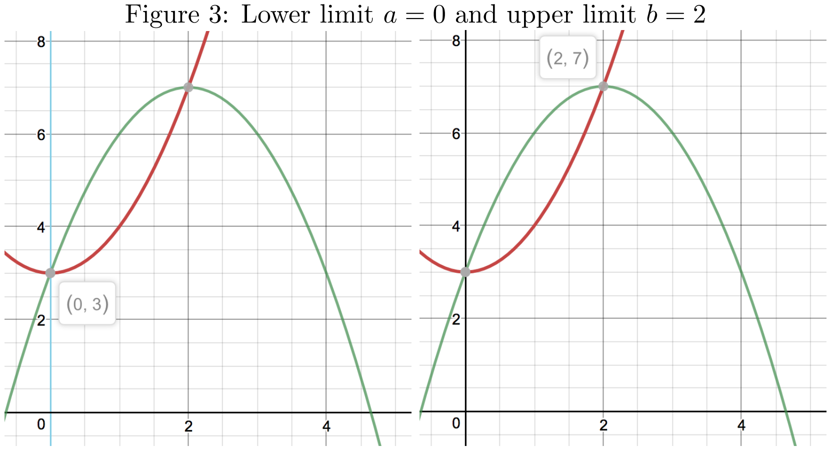
Let $f(x) = -(x-2)^2 + 7$ and $g(x) = x^2 + 3$.

Figure 2: Green is $f(x)$, Red is $g(x)$



The equations and the area they define is shown above (Figure 2). To find the centroid, we want to find the center of mass, or the coordinates $(\bar{x}, \bar{y})$ where $\bar{x}$ is the x coordinate of the centroid and $\bar{y}$ is the y coordinate of the centroid. To do so, let's first find the x coordinates which are the domain of the area between the two curves.

Figure 3: Lower limit $a = 0$ and upper limit $b = 2$



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We can identify from the graphs above (Figure 3) that the lower limit a of the area between the curves is 0 and the upper limit b of the area between the curves is 2 (likewise, we can set $f(x)$ and $g(x)$ equal to each other to arrive at the same solutions $x = 0, 2$).

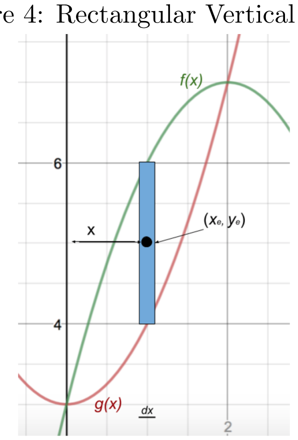
Accordingly, we can find the total area between the curves by integrating with a and b as our lower and upper limits, respectively. Consider the area divided into vertical rectangular strips with width Δx or rather dx when we integrate. Let $f(x) = -(x-2)^2 + 7$ and $g(x) = x^2 + 3$. Then the height of each strip is:

$$\begin{aligned} h &= f(x) - g(x) \\ &= -(x-2)^2 + 7 - (x^2 + 3) \\ &= -x^2 + 4x - 4 + 7 - x^2 - 3 \\ &= -2x^2 + 4x \end{aligned}$$

Therefore $dA = (-2x^2 + 4x)dx$ and we can integrate dA over $(0, 2)$ to find the total area A of the area between the curves:

$$\begin{aligned} A &= \int_0^2 dA \\ &= \int_0^2 (-2x^2 + 4x)dx \\ &= \frac{-2x^3}{3} + 2x^2 \Big|_0^2 \\ &= \frac{8}{3} \end{aligned}$$

Figure 4: Rectangular Vertical Strips



Note from the graph above (Figure 4) that $x_e = x$. Thus the $\bar{x}$ coordinate

Cont.

of the centroid or center of mass coordinates $(\bar{x}, \bar{y})$ is:

$$\begin{aligned}\bar{x} = \frac{\text{total moments}}{\text{total area}} &= \frac{1}{A}\int_a^b x_e(f(x) - g(x))dx \\ &= \frac{1}{A}\int_a^b x(f(x) - g(x))dx \\ &= \frac{1}{A}\int_a^b x dA\end{aligned}$$

As we have already determined the total area A let us first consider the integral for total moments by itself:

$$\begin{aligned}\int_a^b x dA &= \int_0^2 x(-2x^2 + 4x)dx \\ &= \int_0^2 -2x^3 + 4x^2 dx \\ &= \left(\frac{-2x^4}{4} + \frac{4x^3}{3}\right)\Bigg|_0^2 \\ &= \left(\frac{-(2)^4}{2} + \frac{4(2)^3}{3}\right) \\ &= \left(\frac{-16}{2} + \frac{32}{3}\right) \\ &= \frac{8}{3}\end{aligned}$$

Returning to the formula for $\bar{x}$:

$$\bar{x} = \frac{1}{A}\int_a^b x dA = \frac{1}{\frac{8}{3}}\left(\frac{8}{3}\right) = \left(\frac{3}{8}\right)\left(\frac{8}{3}\right) = 1$$

Thus we have determined that $\bar{x} = 1$. Now to consider the $\bar{y}$ coordinate of the centroid $(\bar{x}, \bar{y})$, notice from an evaluation of the graph that

$$y_e = \frac{f(x) + g(x)}{2}$$

Accordingly:

$$\bar{y} = \frac{1}{A}\int_a^b y_e dA = \frac{1}{A}\int_a^b \frac{f(x) + g(x)}{2} dA$$

Cont.

Again to first consider the integral for total moments by itself:

$$\int_a^b \frac{f(x)+g(x)}{2} dA = \int_0^2 \frac{-(x-2)^2+7+(x^2+3)}{2}(-2x^2+4x)dx$$
$$= \int_0^2 (2x+3)(-2x^2+4x)dx$$
$$= \int_0^2 (-4x^3+2x^2+12x)dx$$
$$= -x^4+\frac{2}{3}x^3+6x^2\Big|_0^2$$
$$= -(2)^4+\frac{2}{3}(2)^3+6(2)^2$$
$$= \frac{40}{3}$$

Returning to the formula for $\bar{y}$:

$$\bar{y} = \frac{1}{A}\int_a^b y_e dA = \left(\frac{1}{\frac{8}{3}}\right)\left(\frac{40}{3}\right) = \left(\frac{3}{8}\right)\left(\frac{40}{3}\right) = 5$$

Therefore, for the area defined by the curves $f(x)$ and $g(x)$ the centroid $(\bar{x}, \bar{y}) = (1, 5)$.

3. Please write the following expression using LaTeX formatting:

5 (square root 3x) + 2x^2 – (3x/2)

In LaTeX formatting:

$$5\sqrt{3x} + 2x^2 - \frac{3x}{2}$$

which before compiling appears as:

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$$5\sqrt{3x} + 2x^2 - \frac{3x}{2}$$
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The End.