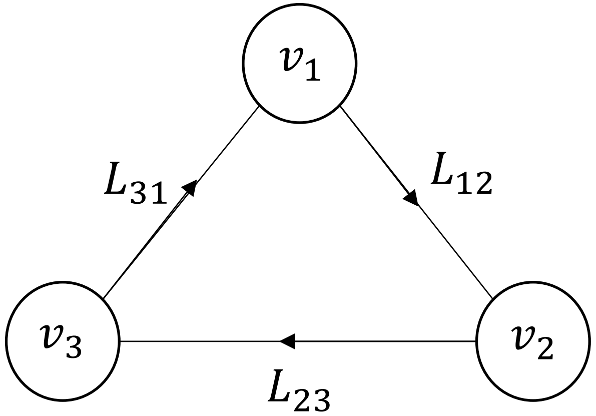
A Symmetric Chromatic Function for Voltage Graphs

Noah Donald
The Ohio State University, Columbus
Undergraduate Research Advisor: Prof. Sergei Chmutov
Assistance on Project by Ishaan Shah
OSU-Marion, MIGHTY LXII

Voltage Graphs and a Proper Coloring.

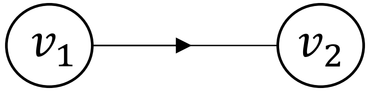
A voltage graph consists of:
- A graph, $G = (V, E \subset V \times V)$
- A group, $\mathcal{H}$
- An edge labeling, L
- An edge polarization, ϵ

Example of a voltage graph
$(G, \mathcal{H}, L, \epsilon)$



$L: E \rightarrow \mathcal{H}$
$\epsilon: E \rightarrow \{-1,1\}$

Example of the edge polarization function:

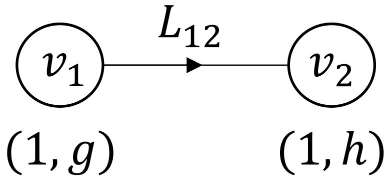


$$\epsilon\big((v_1, v_2)\big) = 1$$
$$\epsilon\big((v_2, v_1)\big) = -1$$

The vertices of the voltage graph are colored using the following functions:
- $\kappa_1: V \rightarrow \mathbb{N}$
- $\kappa_2: V \rightarrow \mathcal{H}$
- $\kappa: V \rightarrow \mathbb{N} \times \mathcal{H}$ where $\kappa(v) = \big(\kappa_1(v), \kappa_2(v)\big)$

An edge is properly colored if any one of the following holds for an edge (v_i, v_j):
1. $\kappa_1(v_i) \neq \kappa_1(v_j)$
2. If $\kappa_1(v_i) = \kappa_1(v_j)$ and $\epsilon\left(\left(v_i, v_j\right)\right) = 1$, then $L\left(\left(v_i, v_j\right)\right)\kappa_2(v_i) \neq \kappa_2(v_j)$
3. If $\kappa_1(v_i) = \kappa_2(v_j)$ and $\epsilon\left(\left(v_i, v_j\right)\right) = -1$, then $L\left(\left(v_i, v_j\right)\right)\kappa_2(v_j) = \kappa_2(v_i)$.

Example of a properly colored edge



$$L_{12}g \neq h$$

A Symmetric Chromatic Function.

Notation:

- $\delta_{ij} = \delta_{\kappa_1(v_i),\kappa_1(v_j)}$
- $\delta^{Lij} = \delta^{L\left((v_i,v_j)\right)\kappa_2(v_i),\kappa_2(v_j)}$
- $\delta^{iLj} = \delta^{\kappa_2(v_i),L\left((v_i,v_j)\right)\kappa_2(v_j)}$
- $(i,j) = (v_i, v_j)$
- $x_\kappa = x_{\kappa(v_1)} x_{\kappa_{v_2}} \dots x_{\kappa(v_n)}, |V| = n$

We define the symmetric chromatic function (SCF) as:

$$\mathrm{X}_{(G,\mathcal{H},L,\epsilon)} = \sum_{\substack{\kappa:V\to\mathbb{N}\times\mathcal{H} \\ \kappa \text{ is proper}}} x_\kappa$$

We can sum over all colorings of the graph by introducing the second factor to “encode” the proper coloring condition.

$$\mathrm{X}_{(G,\mathcal{H},L,\epsilon)} = \sum_{\kappa:V\to\mathbb{N}\times\mathcal{H}} x_\kappa \prod_{(i,j)\in E} (1 - \phi_{ij})$$

Here, ϕ_{ij} is the 0-1 valued term that determines when a coloring is improper.

$$\phi_{ij} = \frac{1}{2}\delta_{ij}\left(\delta^{Lij} + \delta^{iLj} + \epsilon_{ij}\left(\delta^{Lij} - \delta^{iLj}\right)\right)$$

Expanding the product term in the SCF gives a subsets of edges formulation.

$$\mathrm{X}_{(G,\mathcal{H},L,\epsilon)} = \sum_{\kappa:V\to\mathbb{N}\times\mathcal{H}} x_\kappa \sum_{S\subset E} (-1)^{|S|}\phi_{ij}$$

If we have the coloring $\kappa: V \to \mathbb{N} \times \mathcal{H}$ and $Im[\kappa] \subset \mathbb{N} \times \mathcal{H}$, then the coefficient of the term x_κ in the SCF is the number of ways to properly color the graph using only and all of the elements of $Im[\kappa]$.

If G_1, G_2 are disconnected graphs and $G = G_1 \sqcup G_2$, then $\mathrm{X}_G = \mathrm{X}_{G_1}\mathrm{X}_{G_2}$

Example Calculation:

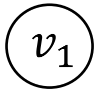
$$v_1 \xrightarrow{L_{12}} v_2$$

$$\mathrm{X} = \left(\sum_{(n,g)\in\mathbb{N}\times\mathcal{H}} x_{(n,g)}\right)^2 - \sum_{(n,g)\in\mathbb{N}\times\mathcal{H}} x_{(n,g)} x_{(n,L_{12}g)}$$

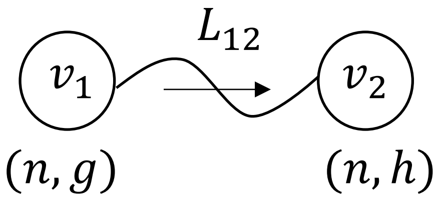
Squiggly Voltage Graphs.

Types of Edges:

- Full edges are edges which require the proper coloring condition for the vertices connected to the edge.
- Squiggly edges are edges which require an improper coloring for the vertices connected to the edge.



Proper coloring of a squiggly edge:

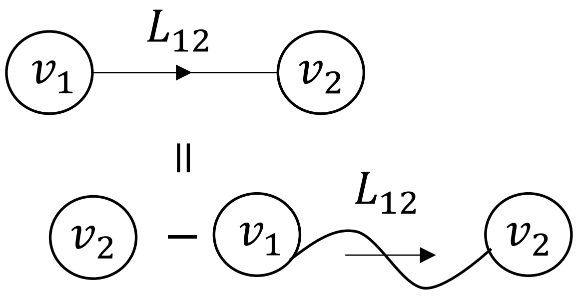


$$L_{12} g = h$$

Turning full edges into squiggly edges: Let $e \in F$ and $E = F \sqcup S$. Then,

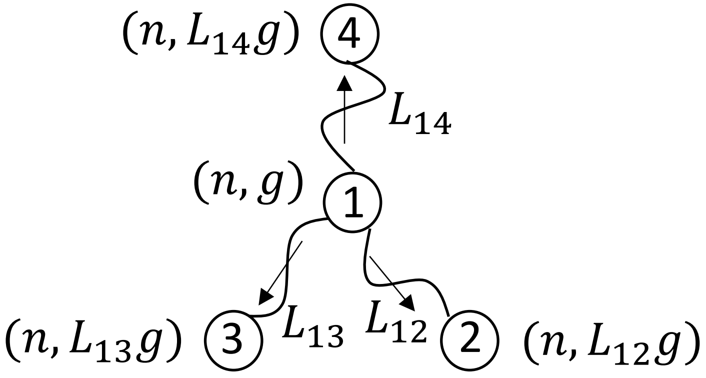
$$\mathrm{X}_{((V, F \sqcup S), \dots)} = \mathrm{X}_{((V, (F \setminus e) \sqcup S), \dots)} - \mathrm{X}_{(V, (F \setminus e) \sqcup (S \sqcup e), \dots)}$$

Example of this formula:

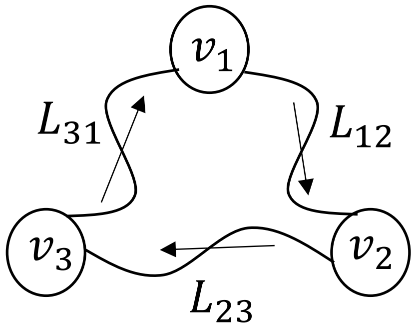


Using this algorithm, we can decompose voltage graphs into combinations of disconnected graphs and connected squiggly graphs.

By coloring any one vertex in a connected squiggly graph, we can find a proper coloring by propagating the group element by paths to any other group element.

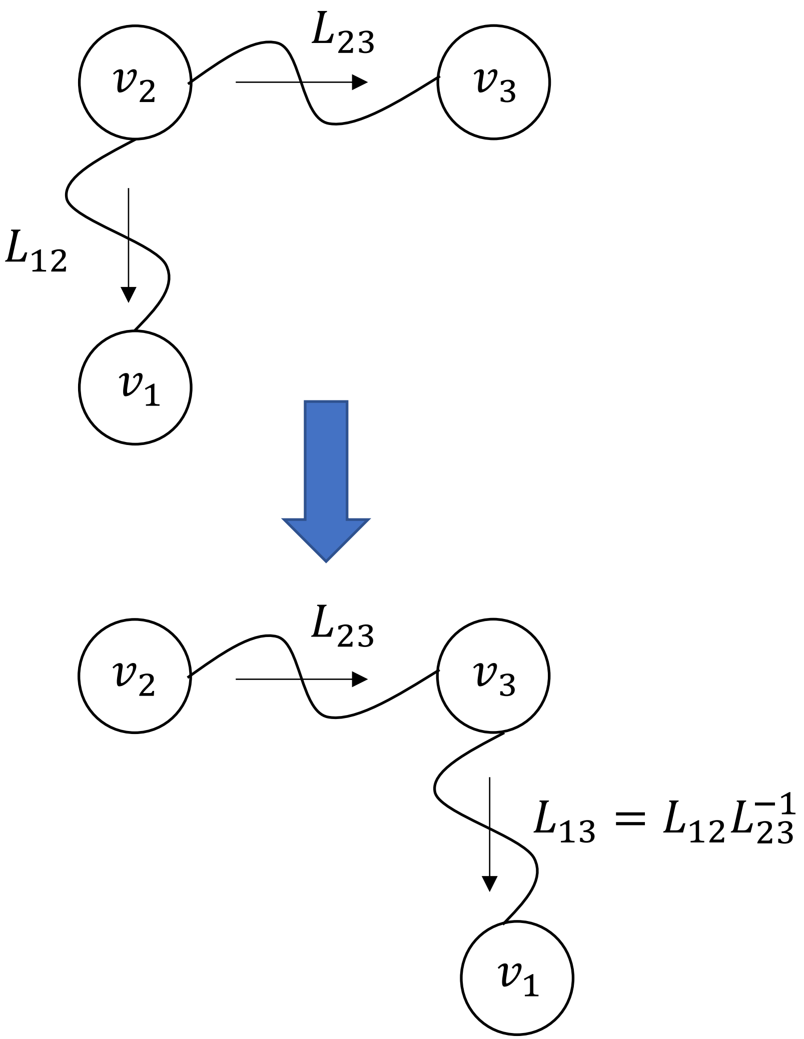


Example of a squiggly graph whose colorablity is independent of the chosen coloring.



For any coloring to work we require: $L_{12} L_{23} L_{31} = e$

Moving Rules.

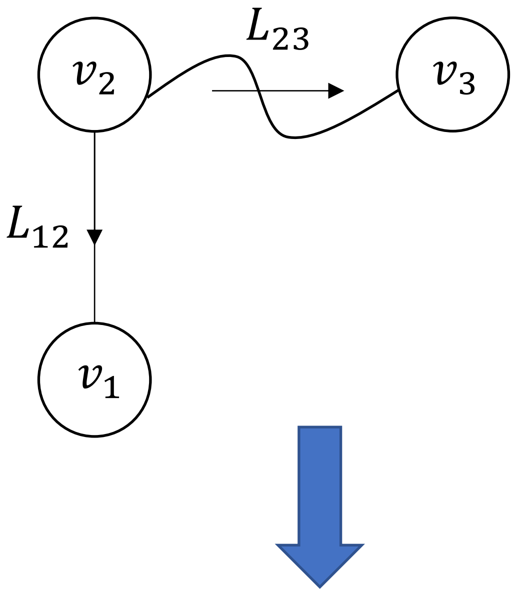


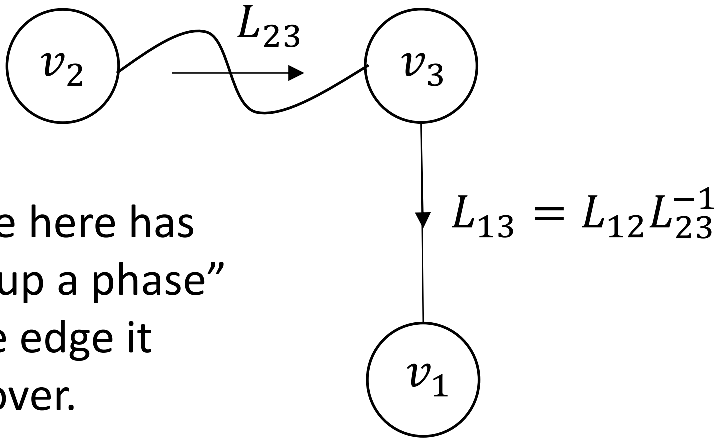
The goal is to move edges over other edges and leave the SCF invariant under this transformation.

If we think of the vertices as labeled here, then we can write a formula which incorporates arbitrary edge polarizations.

$$\boxed{L_{13}^{\epsilon_{13}} = L_{23}^{\epsilon_{23}} L_{12}^{\epsilon_{12}}}$$

This formula holds for moving squiggly edges over squiggly edges and for moving full edges over squiggly edges.





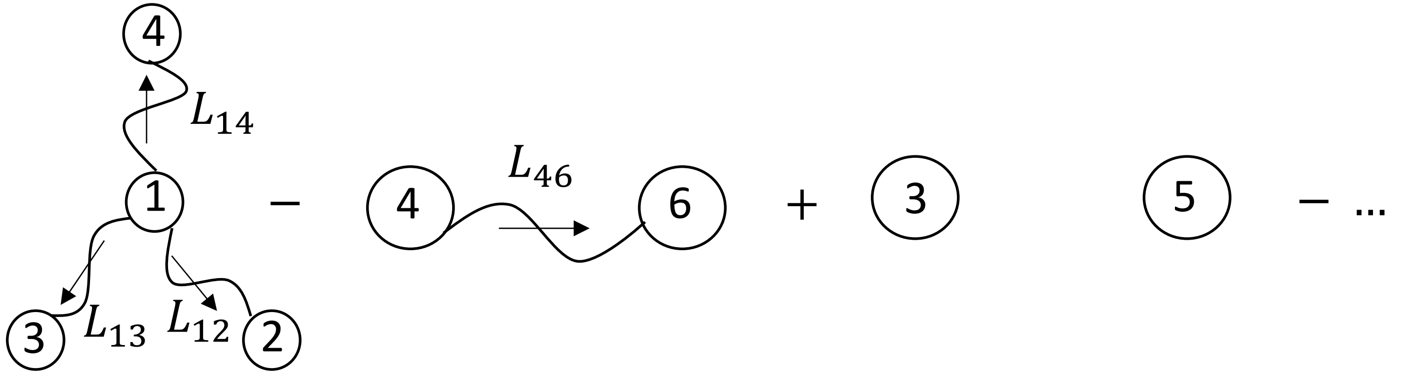
The edge here has “picked up a phase” from the edge it moved over.

Going Forward and Algebraic Bases.

Turning full edges into squiggly edges: $X_{((V,F\sqcup S),\dots)} = X_{((V,(F\setminus e)\sqcup S),\dots)} - X_{(V,(F\setminus e)\sqcup(S\sqcup e),\dots)}$

$(T, \mathcal{H}, L, \epsilon)$ =

(T is a tree)



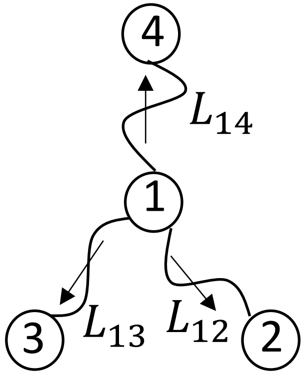
By performing the decomposition into "radiating voltage stars", we can use this as an algebraic basis for the SCF.

- Let $\tau : \mathcal{H} \times \mathcal{H}^n \to \mathcal{H}^n$ be a group action defined by, $\tau\big(g, (h_1, \dots, h_n)\big) = (h_1 g, \dots, h_n g)$.
- $v = \big(e, L_{ij}, L_{ik}, \dots L_{im}\big) \in \mathcal{H}^n$
- $\mathcal{O}_{\mathcal{H}^n}(v)$ is the orbit of v in $\mathcal{H}^n$ under the group action τ.
- $\pi_i : \mathbb{N} \times \mathcal{O}_{\mathcal{H}^n}(v) \to \mathcal{H},\ i \in \{2, \dots, n+1\}$
- $\pi_1 : \mathbb{N} \times \mathcal{O}_{\mathcal{H}^n}(v) \to \mathbb{N}$ defined by
- $\pi_i(k, g_1, \dots, g_n) = \begin{cases} k, i = 1 \\ g_i, i \neq 1 \end{cases}$
- $x_{\pi(\sigma)} = x_{(k,\pi_2(\sigma))} x_{(k,\pi_3(\sigma))} \dots x_{(k,\pi_{n+1}(\sigma))}$

Star Basis Element:

$$S_{v,n} = \sum_{\sigma \in \mathbb{N} \times \mathcal{O}_{\mathcal{H}^n}(v)} x_{\pi(\sigma)}$$

Example term:



$$S_{(e,L_{14},L_{13},L_{12}),4} = \sum_{\sigma \in \mathbb{N} \times \mathcal{O}_{\mathcal{H}^4}((e,L_{14},L_{13},L_{12}))} x_{\pi(\sigma)}$$

References.

- Noble, S., Welsh, D. (1999). A weighted graph polynomial from chromatic invariants of knots. *Annales de l'institut Fourier, 49(3), 1057– 1087.*
- Stanley, R.P. (1995). A Symmetric Function Generalization of the Chromatic Polynomial of a Graph. *Advances in Mathematics, 111, 166-194.*
- Zaslavsky, T. (1982). Signed Graph Coloring. *Discrete Math, 39, 215-228.*
- Zaslavsky, T. (1982). Chromatic invariants of signed graphs. *Discrete Math, 42, 287–312.*