Principal Fiber Bundles and the Minimal Georgi-Glashow Model

By Noah Donald

Introduction

For the past 50 years, the Standard Model of elementary particle physics has provided the most complete picture of the universe to date. It is both a quantum field theory and an $SU(3) \times SU(2) \times U(1)$ gauge theory which contains four gauge bosons, one complex scalar doublet, three generations of Quarks, and three generations of Leptons. Yet, certain questions within this model remain unanswered. For example, why do we see three generations of fermions? Why do the Standard Model particles transform under the representations that they do? Why is there a mass hierarchy among the generations? In 1974, Howard Georgi and Sheldon Glashow attempted to explain why the Standard Model particles must carry the representations that they do.

To do this, they began by looking at the Standard Model gauge group, which is a specific example of a Lie group. In the space (Category) of all connected Lie groups, we want to identify the fundamental building blocks that can be put together through the theory of group extensions to build any other connected Lie group. These fundamental building blocks are called simple Lie groups. Characterized alternatively, the simple Lie groups cannot be decomposed into products of other smaller simple Lie groups. When we put together simple Lie groups, the resulting groups are called only semi-simple Lie groups.

In the Standard Model, the gauge group is a direct product of the simple Lie groups, $U(1)$, $SU(2)$, and $SU(3)$. So, the Standard Model gauge group is only semi-simple. Wouldn't it be elegant though if nature was based on just one of these fundamental building blocks instead of a mixture of three separate ones? This is where Georgi and Glashow introduced the idea of replacing the Standard Model gauge group with the compact, connected, simply connected, simple Lie group $SU(5)$, which is large enough to contain the Standard Model gauge groups as subgroups. The compactness requirement is necessary so that the gauge group will yield finite dimensional unitary representations and the simply connected property is desirable so that the group is "maximal" in a topological sense. Also, the rank of $SU(5)$ is four which equals the rank of the Standard Model gauge group. So, in this sense, $SU(5)$ is a Lie group of minimal size which contains the Standard Model gauge group.

1 Grand Unified Theories and Principal Fiber Bundles

The $SU(5)$ gauge theory is often called a "grand unified theory" in the literature, so it is worth discussing for a moment why this is a fitting label for the theory. In a general gauge theory, one is interested in studying Ehresmann connections on principal fiber bundles over a real differentiable manifold. By choosing a local section of the principal bundle, one is able to obtain a local gauge potential for the connection. This local gauge potential is a set of Lie-algebra valued differential one-forms on the manifold and these one-forms are identified with the classical gauge boson fields. Under global gauge transformations, these one-forms will transform under the adjoint representation of the gauge group. Further suppose now that our manifold is Riemannian. Then, since the action of passing from the tangent bundle to the exterior bundle is functorial, we can lift the inner product induced by the metric tensor on the tangent bundle to the exterior bundle. This provides a way to combine tensors together to yield invariant scalar quantities on the manifold.

However, our local gauge potential is not just a differential one-form, it is also Lie algebra valued. So, we need an inner product on the Lie algebra of our gauge group that is invariant under the adjoint representation. For a simple compact gauge group, the Killing form is the unique inner product on the Lie algebra of this group up to multiplication by a non-zero constant. If this constant is fixed to some value, say, g, then the Killing form has been determined to provide the inner product on the Lie algebra. Combining this inner product on the Lie algebra with the inner product on differential forms, provides a way to create invariant scalar quantities (i.e. an action for a gauge theory). This constant, g, can be identified with the gauge coupling constant for a field theory. If the Lie algebra is semi-simple, then we need to pick a coupling constant on each of the simple pieces that compose the Lie algebra. In a physical sense, the magnitude of these gauge coupling constants characterizes the strength of these gauge fields and can be picked independently of one another since they come from separate simple Lie algebras. In the $SU(5)$ theory, there is just one simple Lie algebra. So, all of the gauge bosons are the result of this single gauge group and we also have just one gauge coupling constant to determine the strength of the gauge fields.

This is why $SU(5)$ gauge theory is called a grand unified theory: All of the gauge fields stem from one Lie group and their interaction strengths are only determined by a single gauge coupling constant. After spontaneous symmetry breaking, the coupling constants will in general evolve differently to low energies according to the renormalization group equations. Thus, grand unified theories predict a unification scale (somewhere near the Planck scale) at which the Standard Model gauge couplings achieve the same value.

The Standard Model Representation

In the Standard Model, the gauge group, $SU(3) \times SU(2) \times U(1)$ is the direct product of three simple, compact Lie groups. To understand the Standard Model representation, we need the following fact from the finite dimensional representation theory of compact Lie groups: Given Lie groups, G and H, and finite dimensional irreducible representations of each of these groups, ρ_G and ρ_H, then the tensor product $\rho_G \otimes \rho_H$ is an irreducible representation of $G \times H$. Moreover, every irreducible representation of $G \times H$ is of this form. So, to understand the irreducible representations of the Standard Model gauge group, we just need to understand the representations of each of its factors individually and then tensor them together in the end. Additionally, we have the obvious fact that taking the direct sum of irreducible representations of a Lie group is also a (reducible) representation of the group. This allows us to put together all of the irreducible representations of the Standard Model gauge group into a single representation. The last piece of information that we will need is the notion of a dual representation. Given a representation, $\rho : G \to \mathrm{GL}(V)$ on a finite dimensional vector space, V, we can define the dual representation $\rho^* : G \to \mathrm{GL}(V^*)$ on the dual space V^* by

$$\rho^*(g) = (\rho(g^{-1}))^T. \tag{1}$$

We will need the dual representation because, if the fundamental fermions of the Standard Model transform under a given irreducible representation, then the corresponding antiparticles will transform under the dual representation. Note that if a representation is irreducible, then its dual is also irreducible, but may not be isomorphic to the original representation.

In this paper, I will simply state the representation that each of the first-generation fermions will transform according to. However, a bit of notation is in order before the representations are listed. Let (y) correspond to the representation of $U(1)$ where y labels the hypercharge of the particle. Let $\mathbf{2}$ denote the fundamental representation of $SU(2)$ and $\mathbf{1}$ denote the trivial representation of $SU(2)$. Finally, let $\mathbf{3}$ denote the fundamental representation of $SU(3)$ and $\mathbf{1}$

denote the trivial representation of $SU(3)$. Then we can tabulate our first-generation fermions and their representations in table 1.

Particle Name	Symbol	Representation
Left-handed Leptons	L	$(\mathbf{1}, \mathbf{2}, -1)$
Left-handed Quarks	Q	$(\mathbf{3}, \mathbf{2}, 1/3)$
Right-handed Neutrino	ν_R	$(\mathbf{1}, \mathbf{1}, 0)$
Right-handed electron	e_R	$(\mathbf{1}, \mathbf{1}, -2)$
Right-handed Up Quark	u_R	$(\mathbf{3}, \mathbf{1}, 4/3)$
Right-handed Down Quark	d_R	$(\mathbf{3}, \mathbf{1}, -2/3)$

Table 1: Table of all Standard Model fermions and their representations including the right-handed neutrino.

Now, we can take the direct sum of each of these representations,

$$F = (\mathbf{2}, \mathbf{1}, -1/2) \oplus (\mathbf{2}, \mathbf{3}, 1/6) \oplus (\mathbf{1}, \mathbf{1}, 0) \oplus (\mathbf{1}, \mathbf{1}, -1) \oplus (\mathbf{1}, \mathbf{3}, 2/3) \oplus (\mathbf{1}, \mathbf{3}, -1/3), \tag{2}$$

which we will call the Fermion representation. But since we also have the antiparticles, we need the anti-Fermion representation, $\overline{F}$ which is the same as F but with each summand replaced with its dual representation. So, the total representation space of all of the first-generation fermions is given by

$$F \oplus \overline{F} \tag{3}$$

which we will call the Standard Model representation. For all three known generations of Fermions, we will have three separate copies of $F \oplus \overline{F}$, however we will just focus on the first generation for the rest of this paper.

The $SU(5)$ Representation

The Standard Model representation is interesting because it's $\mathbb{C}$-dimension is $16 + 16 = 32$. Thus, we can label each dimension of the Standard Model representation by a five-digit binary code. Each "1" or "0" in this code will correspond to a "yes" or "no" answer to each of the following five questions about the particle: Is the particle isospin up? Is it isospin down? Is it red? Is it green? Is it blue? For example, suppose our code is 01100. This implies that the particle we are looking for has isospin down and it is red. Thus, this particle must be the left-handed red down quark. Notice that if we take any five-digit code and swap each 1 for a 0 and likewise each 0 for a 1, then we get the antiparticle of the particle whose code we swapped. So, in our example, we would have the code 10011 which is an isospin up particle whose color is a mixture of blue and green. Thus, we recognize mixtures of blue and green as anti-red. So, we have a right-handed anti-red anti-up quark. In addition, we see that Leptons are "white" with code ending in 111, anti-Leptons are "black" with code ending in 000, quarks have just one color, and antiquarks have two colors.

Since this binary representation is a perfectly valid way to identify the Standard Model fermions, lets take each of these properties and make it a basis vector in the vector space $\mathbb{C}^5$. So, we have u, d, r, g, b as basis vectors where here u stands for isospin up, d stands for isospin down, and r, g, b stands for each respective color. Notice that $\mathbb{C}^5$ is the fundamental representation space of the group $SU(5)$ once $\mathbb{C}^5$ is endowed with the structure of the complex inner product. But particles, as we have seen, are specific combinations of these basis vectors. So, consider the exterior algebra of this space, $\Lambda(\mathbb{C}^5)$. Notice that the dimension of this space is also $2^5 = 32$. Also, notice that the map $\Lambda : V \to \Lambda(V)$ is a covariant functor from the category of complex vector spaces to itself. So, the fundamental representation of $SU(5)$ on $\mathbb{C}^5$ lifts to a representation on $\Lambda(\mathbb{C}^5)$ which is reducible and decomposes into irreducibles via its natural $\mathbb{Z}$-grading.

Lets look at the original space $\mathbb{C}^5$ once more though. Clearly the first two properties are related to each other and the last three properties are related to each other. So, consider the vector space decomposition $\mathbb{C}^5 = \mathbb{C}^2 \oplus \mathbb{C}^3$ and only consider the subgroup of $SU(5)$ in the fundamental representation which preserves this splitting of $\mathbb{C}^5$ into the isospin part and the color part. The subgroup that has this property is $S(U(2)\times U(3))$. Since it is useful to consider the action of this subgroup on $\mathbb{C}^5$ in the fundamental representation, we can consider the action of this subgroup on the space $\Lambda(\mathbb{C}^5)$. Amazingly, it turns out that, as a representation, this restricted representation of $SU(5)$ to $S(U(2) \times U(3))$ on the space $\Lambda(\mathbb{C}^5)$ is isomorphic to the Standard Model representation, $F \oplus \overline{F}$. Table 2 briefly summarizes the way the representation decomposes:

$\Lambda^k(\mathbb{C}^5)$	$S(U(2) \times U(3))$ irreducible reps.
$\Lambda^0(\mathbb{C}^5)$	$(\mathbf{1},\mathbf{1})$
$\Lambda^1(\mathbb{C}^5)$	$(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$
$\Lambda^2(\mathbb{C}^5)$	$(\mathbf{1},\mathbf{3}) \oplus (\mathbf{2},\mathbf{3})$
$\Lambda^3(\mathbb{C}^5)$	$(\mathbf{2},\overline{\mathbf{3}}) \oplus (\mathbf{1},\overline{\mathbf{3}})$
$\Lambda^4(\mathbb{C}^5)$	$(\mathbf{1},\overline{\mathbf{3}}) \oplus (\mathbf{2},\mathbf{1})$
$\Lambda^5(\mathbb{C}^5)$	$(\mathbf{1},\mathbf{1})$

Table 2: The decomposition of each vector space in the grading of $\Lambda(\mathbb{C}^5)$ into irreducible representations

Notice that the left-handed particles show up in the even grading of $\Lambda(\mathbb{C}^5)$ while the right-handed particles show up in the odd grading of $\Lambda(\mathbb{C}^5)$. It is interesting to point out that the even grading forms a subalgebra of the exterior algebra while the odd grading does not. Thus, we can see algebraically that left-handed particles and right-handed particles are fundamentally different. Also note that the Hodge star operator $* : \Lambda^k(\mathbb{C}^5) \to \Lambda^{5-k}(\mathbb{C}^5)$ allows one to change between particles and antiparticles. In terms of the binary representation of particles, the Hodge star is the operation that swaps "0"s and "1"s in a particles code. Note, however, that this is not an intertwining of representations under the broken gauge group. It is also interesting to see how this representation almost beckons for the existence of right-handed neutrinos. So, while the Georgi-Glashow model was originally proposed with massless neutrinos, it is possible within this framework to incorporate a see-saw type mechanism to generate neutrino masses.

Lets think for a moment about what all of this means. It means that the seemingly arbitrary combinations of hypercharge, weak isospin, and color of the Standard Model fermions are not at all arbitrary, but are required to be exactly what they are due to the $SU(5)$ symmetry. It is also interesting to note that we started by choosing a splitting in the $\mathbb{C}^5$ space between the isospin property and the color property and in the end learned something about hypercharge. It should also be possible to do a similar analysis by beginning with any of the two gauge groups and then using representation theory to learn about the third gauge group. This should give a kind of 2 out of 3 property for studying how particles will transform under their gauge groups. As Lie groups, the group $S(U(3) \times U(2))$ is isomorphic to the Standard Model gauge group $(SU(2) \times SU(3) \times U(1))/\mathbb{Z}_6$ modulo a discrete normal subgroup. It can be shown by just using representation theory, that this discrete normal subgroup of the Standard Model gauge group is $\mathbb{Z}_6$ and it acts trivially on the fundamental fermions. So, we could claim that $S(U(2) \times U(3))$ should be the correct gauge group of the Standard Model.

The Gauge Bosons

The real dimension of the complexified Lie algebra of $SU(5)$ is 24 and gauge theory dictates that the gauge bosons of the model must transform under the adjoint representation of $SU(5)$. Since $SU(5)$ is a simple Lie algebra, this representation is irreducible. Thus, in the unbroken theory 24 massless gauge fields are predicted. However, after symmetry breaking, this representation is restricted to the Standard Model gauge group. The adjoint representation is then reducible under this restriction. The decomposition of this adjoint representation into irreducible representations is as follows:

$$(\mathbf{8}, \mathbf{1}, 0) \oplus (\mathbf{1}, \mathbf{3}, 0) \oplus (\mathbf{1}, \mathbf{1}, 0) \oplus (\mathbf{3}, \mathbf{2}, -5/6) \oplus (\overline{\mathbf{3}}, \mathbf{2}, 5/6). \tag{4}$$

The first index lists the representation of $SU(2)$ while the second index lists the representation of $SU(3)$. The third index denotes the hypercharge. The first factor can be recognized as the eight gluons of the Standard Model, while the second factor are the W bosons, and the third factor is the B-boson. The remaining two factors are the predicted X and Y bosons in the $SU(5)$ model and these bosons are massive since they result from the broken piece of the $SU(5)$ gauge symmetry. Note that the dimension of the representation of the X and Y bosons is 6 and that they carry both flavor and color symmetry. Since they are also the remnants of the broken, unified, theory of fermions, their interactions will violate both lepton number and baryon number conservation.

Spontaneous Symmetry Breaking

Lets consider the spontaneous symmetry breaking from the $SU(5)$ group down to the Standard Model gauge group. The mathematical formalism that gives way to symmetry breaking in the context of principal fiber bundles is called structure group reduction. One way of performing this reduction is by identifying a cross section for each of the representations giving rise to associated vector bundles over spacetime. The two representations that we have considered in this paper are the adjoint representation and the fundamental representation. So, we will look at these two cases separately. We will identify classical Higgs fields as fields taking values in these cross sections. When these classical Higgs fields are included and the Lagrangian formalism is applied, vacuum expectation values arise naturally. By analyzing the symmetries of each of the possible vacuum expectation values, we can study the possible breaking patterns from $SU(5)$ down to various subgroups. Physical fields are understood as those which fluctuate about the classical vacuum. So, by expanding the fields about this vacuum expectation value in the Lagrangian formalism we identify the physical fields which leads to the familiar mass generating Higgs mechanism of the Standard Model. We will not attempt to do all of this in this section, however we will look at the identification of cross sections and briefly discuss vacuum expectation values.

First, consider the adjoint representation of $SU(5)$ on its complexified Lie algebra. We wish to find a cross section for this representation. Recall that a cross section for a representation is a subset of the representation space that intersects the orbit of each element precisely once. We now need the following fact: every orbit of the adjoint representation intersects the Cartan subalgebra. The dimension of the Cartan subalgebra is equal to the rank of the Lie group which is equal to four for $SU(5)$. So, every element in the 24-dimensional Lie algebra eventually intersects this particular four-dimensional subspace. We can now use another important fact: The complexification of the Lie algebra $su(5)$ is isomorphic to the Lie algebra $sl(5, \mathbb{C})$. So, we can represent each element in the Lie algebra as a 5×5 traceless matrix. Note that the Cartan subalgebra consists of all such matricies which are diagonal with purely imaginary entries. Another important fact is the following: Every orbit in the Lie algebra intersects the Cartan subalgebra precisely in the orbit of an element in the Cartan subalgebra under the action of the Weyl group. For $SU(5)$, the Weyl group is the finite symmetric group on five characters, S_5. In the representation of the Cartan subalgebra by 5×5 traceless purely imaginary matrices, the Weyl group simply acts by permuting the diagonal entries of the matrix. So, to pick a representative from each Weyl orbit, we only consider the matrices with non-increasing elements down the diagonal (note that the set $i\mathbb{R}$ allows for this total ordering). Thus, elements of this type in our Cartan subalgebra constitute a cross section for the adjoint representation. Note that this cross section contains the zero vector, is closed under multiplication by non-negative real numbers, and is closed under addition. So geometrically, the cross section looks like a half plane. The subgroup of $SU(5)$ which preserves the entire cross section (called an isotropy subgroup) is the Lie group that is the exponential of the Cartan subalgebra (called a maximal torus). Any kind of map on spacetime that takes values in this cross section can be recognized as a classical Higgs field.

Now in the Lagrangian formalism, we can form a Higgs potential for this field which will contain some arbitrary real couplings. Depending on the sign of these couplings, we can find a minimum of the potential as a subset of the cross section. If this minimum were just a point in the cross section, this point (which is called a vacuum expectation value) can be written as a scalar multiple of some diagonal traceless matrix with our chosen conventions. In general, this vacuum expectation value can have a larger isotropy subgroup than the one which preserves the entire cross section. Note though that to get a larger isotropy subgroup one needs to have a symmetry in the vacuum expectation value. This translates to the fact that we need to pick some values in our vacuum expectation value equal to each other. This leads to several different cases that each determine a different spontaneous symmetry breaking pattern. Up to equivalence of isotropy groups, we can have the top four diagonal elements being equal, the top three diagonal elements being equal, the top two diagonal elements being equal and the rest different, or the top two diagonal elements being equal and next two diagonal elements also being equal. Each of these will lead to either a breaking of $SU(5)$ to $(SU(4) \times U(1))/\mathbb{Z}_4$ or to $S(U(3) \times U(2))$. One example of breaking to the Standard Model gauge group is when the top two are equal and the next two are also equal. Suppose also that the bottom entry is equal to the two above it. Generically, we can write the diagonal of this matrix as $i\ \mathrm{diag}(a, a, b, b, b)$. So, we have that $2a + 3b = 0$. So, $b = -\frac{2}{3}a$. If $a = \frac{1}{2}$, then $b = -\frac{1}{3}$. Thus, a vacuum expectation value proportional to this matrix will provide a breaking of $SU(5)$ to the Standard Model gauge group.

For further symmetry breaking from the Standard Model gauge group down to the gauge group $SU(3)_c \times U(1)_Q$, we study the fundamental representation of $SU(5)$. It is much easier to find a cross section for this representation. We can simply choose any ray extending outward from the origin. Recall that we have a natural splitting of this space as $\mathbb{C}^5 = \mathbb{C}^2 \oplus \mathbb{C}^3$ where we identify the $\mathbb{C}^2$ piece as the span of the isospin up and isospin down states. So, any vector in this cross section is of the form,

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{5}$$

Here, we identify the ϕ_2 as living in the $\mathbb{C}^2$ subspace and the ϕ_3 as living in the $\mathbb{C}^3$ subspace. This ϕ_2 piece of the Higgs field is identified with the Standard Model Higgs as an $SU(2)$ doublet and the ϕ_3 piece is identified as a new $SU(3)$ triplet which remains untouched when ϕ_2 acquires a vacuum expectation value in the $\mathbb{C}^2$ subspace. This definition of the Higgs leads to problems though when considering its interaction with the adjoint representation Higgs field of the previous section. The problem is called the doublet-triplet splitting problem and it requires a lot of fine tuning in the model to get the masses of these new $SU(3)$-particles to be consistent with experimental proton-decay measurements.

Conclusion

The Georgi-Glashow model has had both experimental and theoretical success, but also has some shortcomings. For example, the $SU(5)$ model can accurately predict the Weinberg mixing angle. Also, as was discussed earlier, the hypercharges of all the first-generation fermions can be predicted. The model also provides a mechanism for proton decay through the mediation of an X boson. The predicted lifetime of a proton is then inversely proportional to the mass squared of the X boson. However, the predicted mass of this particle is too small. The proton lifetime predicted by this model has been ruled out experimentally by detectors like Super-Kamiokande. However, this minimal $SU(5)$ model has spurred many other grand unified theories to be developed. This includes theories like Pati-Salam model, flipped $SU(5)$, supersymmetric $SU(5)$, and $SO(10)$. Some of these theories have not yet been ruled out experimentally. In fact, the Pati-Salam model and the minimal $SU(5)$ model can be viewed as two different routes toward the $SO(10)$ model. So, these models are all closely related to each other and further experimentation in the future will help to either confirm or rule out these other grand unified theories.

References

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