Classical Gauge Theory and Electromagnetism

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Introduction

For the past 50 years, the Standard Model of elementary particle physics has provided the most accurate depiction of the universe to date. It is based on the framework of Quantum Field Theory which postulates the global Poincare invariance of fundamental particles. However, the Standard Model additionally postulates a local $SU(3) \times SU(2) \times U(1)$ gauge symmetry. It is this local symmetry that gives rise to the familiar gauge bosons and assigns charges to the fermion content of the Standard Model. Thus, it appears to be of foundational importance to understand mathematically how gauge theory can feel right at home in the differential geometry of spacetime manifolds. In this paper, I will first build up some of the mathematical objects that are useful to understanding a general gauge theory. I will then take a special look at the case of a $U(1)$ structure group and draw connections to the theory of electromagnetism and Maxwell's equations.

Fiber Bundles

To begin the analysis, suppose we have an n-manifold, M. As was shown in my lecture, it is extremely useful to consider the collection of tangent spaces to M. If $p \in M$, then the tangent space associated to this point is the real vector space T_pM and it also has dimension n. The collection of all the tangent spaces together is denoted by TM and note that it also can be given the structure of a smooth manifold. This notion of "attaching" a mathematical object, like a vector space, to each point on the manifold can be made precise through the language of fiber bundles. In essence, a fiber bundle consists of three mathematical objects, E, M, and F packaged together via a projection map $\pi : E \to M$ such that for each $p \in M$, $\pi^{-1}(p) \cong F$. The space E is called the total space while M is called the base space and F is called the fiber. Note that the symbol $\cong$ is used and not $=$ here. This is to show that the preimage of each point in M looks like a copy of F, but is not exactly the "same" space for different points in M. So, for our example of the tangent spaces, we have the manifold M as our base space, TM as our total space, and an n-dimensional vector space $\mathbb{R}^n$ as our generic fiber. We identify the fiber lying over the point p to be the space T_pM. In the case that the fiber of the fiber bundle has a vector space structure, it is common to refer to this object as a vector bundle. We will also require one more condition for our fiber bundle that will be consequential later on. We will require that for any neighborhood U on M, there exists a continuous and differentiable map $\phi : \pi^{-1}(U) \to U \times F$. This condition basically states that we require our total space to look locally like a product manifold. This condition should be reminiscent of the requirement that our base manifold, M, look locally like $\mathbb{R}^n$. However, the global structure of our total space does not need to look like $M \times F$ in the same way that our n-dimensional manifold, M, does not globally need to look like $\mathbb{R}^n$.

As an example, consider our base space to be the circle S^1 and our fiber to be the closed interval $[-1, 1]$. We can think of attaching this fiber in a smooth manner to S^1 in a few different ways. The most obvious choice is to attach it in the same orientation to each point in S^1, but this will just give something that looks like a cylinder for our total space. Another choice, is to give the fiber a bit of a rotation at each point of S^1. In this case, we end up with a Mobius strip. In the first case, our total space is just the product manifold $S^1 \times [-1, 1]$, however, in the second case it is definitely not.

Lie Groups and Principal Fiber Bundles

Before transitioning from general fiber bundles to principal fiber bundles, a few remarks on Lie groups are in order. A Lie group is an abstract group whose set of elements also carries the structure of a smooth manifold. In addition, the group multiplication map is required to be a smooth map in order for the group structure and the manifold structure to be compatible. It is natural to consider Lie groups as the symmetry groups of other spaces. For example, the Lie group of rotations in three-dimensional space is a symmetry group of the unit sphere. In this way, we can think of the Lie group elements "acting" or "transforming" some other object. In particular, if we forget about the group structure of the Lie group and just view it as a manifold, then the Lie group will act naturally on this manifold. This is totally analogous to how the lie group $U(1)$ (which looks like a circle) acts by rotations on the unit circle S^1. We can also view the Lie algebra, $\mathfrak{g}$, of a Lie group G as the tangent space to G at the identity element, e. So, $\mathfrak{g} = T_eG$. Given a basis of the Lie algebra, we can use the smooth multiplication maps to push these basis vectors forward to a set of linearly independent vector fields on the underlying manifold.

There exists a distinguished Lie-algebra valued differential form called the Mauer-Cartan form of the Lie group. This form is most easily defined in terms of a basis, $\{X_i\}$ of $\mathfrak{g}$ and the dual basis $\{\omega^i\}$ of differential forms in $\Omega^1(G)$. Then, the Mauer-Cartan form, $\theta : \Gamma^1(G) \to \mathfrak{g}$, is defined by

$$\theta(Y) = \sum_i \omega^i(Y) \otimes X_i. \tag{1}$$

Here, we recall that $\Gamma^1(G)$ is the space of smooth vector fields on G. The Mauer-Cartan form will pop-up later, since it's theoretical use is to provide an infinitesimal characterization of the spaces on which these Lie groups act as natural symmetry groups. There also exists a natural map between the Lie algebra and the Lie group given by the exponential map, $\exp : \mathfrak{g} \to G$, which in the case of a matrix group is just given by the Taylor series expansion of the exponential function.

Suppose that we let the fibers of our fiber bundle be the manifold part, F, of a Lie group, G. Then G acts on F such that every element of F "moves" under the action of a non-identity element of G and such that any element $f \in F$ can be transformed to an element $f' \in F$ by a suitable $g \in G$. So, $f' = f \cdot g$ where we choose G to act on elements of F on the right. A fiber bundle of this type is called a principal fiber bundle with structure group G or a G-bundle for short. In theoretical physics, the structure group, G, is also called the gauge group. If the base manifold, M, has dimension n and if the Lie group has dimension m, then the dimension of the total space for this principal fiber bundle will be $n + m$. It is important to recall here that if we have smooth fibers over a smooth manifold, then the total space can also be given the structure of a smooth manifold. Thus, each tangent space to the total space will also have dimension $n+m$. Also, since the fiber over each point of our base manifold is the manifold part of the Lie group, G, we must have that the tangent space to each fiber over some point in the base space is a vector space of dimension m. So, we can view this tangent space to each fiber as an m-dimensional subspace of the tangent space to the total space. This subspace is called the vertical subspace of the G-bundle and note that it can be identified with the Lie algebra of the structure group, $\mathfrak{g}$.

We can choose this identification in a natural way. Let P denote the total space of our G-bundle and let $p \in F_u \subset P$ for some $u \in M$. Consider the map $\sigma : \mathfrak{g} \to T_pF_u$ given by

$$\sigma(X) = \frac{d}{dt}[p \cdot exp(tX)]|_{t=0}. \tag{2}$$

It is useful to extend this map to all of P by viewing T_pF_u as the vertical subspace of T_pP. So, we consider the map $\sigma(p) : \mathfrak{g} \to T_pP$ defined by

$$\sigma(p)(X) = \frac{d}{dt}[p \cdot exp(tX)]|_{t=0}. \tag{3}$$

Now, let $X \in \mathfrak{g}$ be fixed. Then we can construct the fundamental vector field generated by X to be $\sigma(X) : P \to TP$ by leaving the point, p, as a variable in the above equation. Note that by our construction, the fundamental vector field is contained entirely in the vertical subspace. These fundamental vector fields will play a pivotal role in our understanding of connections in principal fiber bundles.

Connections and Principal Fiber Bundles

When we constructed our G-bundle, we were able to explicitly construct the vertical subspace, V_uP, which had dimension, m, while the total tangent space, T_uP, had dimension $n+m$. We would like to decompose T_uP into the direct sum of two subspaces,

$$T_uP = V_uP \oplus H_uP \tag{4}$$

where H_uP is called the horizontal subspace. Note that this is necessarily an n-dimensional subspace. Now, there isn't any natural choice of H_uP like there was for V_uP. In fact, there are infinitely many choices for H_uP to satisfy this vector space decomposition. So, we have to smoothly choose one particular H_uP for each point $u \in P$ by introducing more structure to our manifold. This new structure is called an Ehresmann connection or connection-form. We will define a connection form as an object $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ which is most easily viewed as a map $\omega : T_uP \to \mathfrak{g}$ for each $u \in P$ such that $\omega(\sigma(X)(u)) = X$. Thus, ω is pointwise a left inverse for the fundamental vector field associated with each element of the Lie algebra. To illustrate the most important property of this connection, suppose we are given a generic vector $w \in T_uP$. Then $w = w^h + w^v$ where w_v is the component of w in the vertical subspace and w^h is the component of w in the horizontal subspace. Given the left inverse property for ω, we have $\omega(w) = \omega(w^v)$. So, the subspace of vectors in T_uP that get mapped by ω to $0 \in \mathfrak{g}$ (denoted $ker(\omega)$) is an n-dimensional subspace that we will call H_uP. In this way, for a smooth connection form ω, we can smoothly pick for each $u \in P$ an n-dimensional subspace H_uP of T_uP such that $T_uP = H_uP \oplus V_uP$. Thus, the choice of a connection one-form is equivalent to the choice of a horizontal vector subspace in each tangent space of the total space of our bundle.

There exists a nice formula for projecting a general vector field, Y, on P to a horizontal vector field, Y^h, on P. To write this formula, suppose our Lie algebra has a basis given by the set $\{X_i\}$ and our connection one-form is decomposed as $\omega = \sum_i \omega^i \otimes X_i$ for differential one-forms ω^i on P. Then,

$$Y^h = Y - \sum_i \omega^i(Y)\sigma(X_i) \tag{5}$$

is our desired decomposition since then the vertical component is

$$Y^v = Y - Y^h = \sum_i \omega^i(Y)\sigma(X_i). \tag{6}$$

In a typical introduction to differential geometry, a connection is usually thought of as a way to "connect" nearby tangent spaces. So, while our definition of an Ehresmann connection may seem geometrically rich as a pointwise vector space decomposition, it loses some of this classical flavor for connecting nearby fibers. To remedy this, I will introduce the concept of horizontal path lifting. What this concept aims to do is lift a path $\gamma : [0,1] \to M$ in the base space, M, to a path $\tilde{\gamma} : [0,1] \to P$ in the total space P such that if π is the projection map, then $\pi \circ \tilde{\gamma} = \gamma$. We will also require that $\tilde{\gamma}'(t) \in H_{\tilde{\gamma}(t)}P$ for all t. There is a uniqueness and existence theorem which states that if $\gamma : [0,1] \to M$ is a path in M and $u_0 \in P$ such that $\pi(u_0) = \gamma(0)$, then there exists a unique horizontal path lift $\tilde{\gamma} : [0,1] \to P$ such that $\tilde{\gamma}(0) = u_0$. So, if we have a path γ in M with $\gamma(0) = p$ and $\gamma(1) = q$, then we can uniquely map elements of $\pi^{-1}(p)$ to elements of $\pi^{-1}(q)$. Specifically, if $u \in \pi^{-1}(p)$, then we can construct the horizontal path lift $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = u$ and then map u to $\tilde{\gamma}(1)$. It is important to note that this horizontal lift is path dependent. So, if two different paths in the base space have the same start and end points and are horizontally lifted to paths in the total space with the same initial point, then it is not always true that the lifted paths will have the same end point. I will only mention it in passing here, but the degree to which these path lifts differ in their endpoints is heavily related to the idea of curvature in a principal fiber bundle through the Ambrose-Singer Theorem.

Studying the space of connections on a manifold is a valuable topic in theoretical physics. In the functional integral approach to quantum field theory, integrals are regularly performed over this space. This space is often difficult to characterize however and one is usually not interested in the full space of connections. The space of physically distinct field configurations corresponds to the full space of connections modulo gauge orbits. Indeed, this is the space that is interesting in physics and many Lagrangian approaches, such as that due to Fadeev and Popov, have been utilized to integrate over it.

Gauge Transformations, Curvature, and the Gauge Covariant Derivative

When I first introduced the idea of an Ehresmann connection, I stated that the connection one-form ω is an element of the space $\Omega^1(P, \mathfrak{g})$. While it has been fine so far to view the connection one-form as a Lie algebra-valued one-form on the total space, it would be more gratifying to view the connection form as a Lie algebra-valued one-form on the base space. To do this we need the idea of a pullback map. In general, if $f : N \to M$ is a smooth map between manifolds N and M and f_* is the Jacobian of the map, then the pullback $f^* : \Omega^1(M) \to \Omega^1(N)$ is defined by

$$(f^*\alpha)(X_1, \ldots, X_k) = \alpha(f_*X_1, \ldots, f_*X_k) \tag{7}$$

where α is a k-form on M and X_i are vector fields on N. Now going back to our G-bundle formalism, let U be some local neighborhood of a point $p \in M$ and recall that the set $\Gamma(U, P)$ denotes the set of all smooth maps (called local sections) $s : U \to P$ such that $\pi \circ s = \mathrm{id}_U$. Just like a choice of gauge in theoretical physics, lets choose a $s \in \Gamma(U, P)$ and consider $s^*\omega \in \Omega^1(U) \otimes \mathfrak{g}$. This object is called a local gauge potential for the connection one-form, ω. It is important to note that no information is lost when applying the pullback map. In fact, it is totally possible to reconstruct ω if one is given a gauge potential and a local section. It is also important to realize that we only chose a local section when constructing the local gauge potential. In general, a global section for a principal fiber bundle may or may not exist. So, a local section is often the best we can do. Since this local gauge potential depends on the choice of s, we would like to identify how it changes when we change local sections. Explicitly, if $s, s' \in \Gamma(U, P)$, we would like to find a transformation law between $s^*\omega$ and $(s')^*\omega$. To do this, let $s, s' \in \Gamma(U, P)$ and consider the elements $s(x), s'(x) \in \pi^{-1}(x)$. By our earlier definition of how the structure group, G, acts on the fibers, we know that there must exist a group element $g(x) \in G$ such that $s'(x) = s(x)g(x)$. So, we consider the map $g : U \to G$ that satisfies this requirement at each $x \in U$. Then

$$(s')^*\omega = \mathrm{Ad}_{g^{-1}}(s^*\omega) + g^*\theta \tag{8}$$

where $\mathrm{Ad} : G \to Aut(g)$ is the adjoint representation of the structure group and θ is the Mauer-Cartan form on the structure group and note that $g^* : \Omega^1(G) \to \Omega^1(U)$. If the structure group is a matrix group and we make the identification $g(u) \to g_{ij}(U) \in Mat_n(\mathbb{R})$ in some basis, then

$$(s')^*\omega = g^{-1}(s^*\omega)g + g^{-1}dg \tag{9}$$

where $dg = (dg_{ij})$ for each real (or complex), smooth, scalar function g_{ij} and d is the exterior derivative on our base manifold. This transformation law is the generalized gauge transformation that is familiar in theoretical physics.

As a first example of this, consider a four-dimensional manifold as the base space and let the fiber lying over each point be the space of all ordered bases for the tangent space at each point. Thus, we can think of a point in each fiber as a choice of basis for the tangent space. This fiber is acted upon in the required way by the group $\mathrm{GL}_4(\mathbb{R})$. So, a group element just transforms an oriented basis into a new oriented basis. This $\mathrm{GL}_4(\mathbb{R})$-bundle is usually referred to as the frame bundle. The Lie algebra of $\mathrm{GL}_4(\mathbb{R})$ is the space $\mathfrak{gl}_4(\mathbb{R})$ and clearly this is a matrix group. Now, let U be a local neighborhood on the base manifold and pick two different charts $x, y : U \to \mathbb{R}^4$. Then in U, we have the bases $\{\partial/\partial x^i\}$ and $\{\partial/\partial y^i\}$. Thus, a choice of chart, gives rise to a choice of section in this frame bundle. The group element that transforms between these two bases is just the Jacobian matrix of the coordinate transformation. Now, let ω be an arbitrary connection on the frame bundle (often times the connection is called a Cartan connection in this context), let $A_x = x^*\omega = A^\alpha_{\mu\beta} dx^\mu$ be the local gauge potential induced by the chart x, and let $B = y^*\omega = B^\rho_{\nu\sigma} dy^\nu$ be the local gauge potential induced by the chart y. Then

$$B^\rho_{\nu\sigma} = \frac{\partial y^\rho}{\partial x^\alpha}\frac{\partial x^\beta}{\partial y^\sigma}\frac{\partial x^\mu}{\partial y^\nu} A^\alpha_{\mu\beta} + \frac{\partial y^\rho}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial y^\nu \partial y^\sigma} \tag{10}$$

after applying our formula for a local gauge transformation. This is exactly the same transformation though as the Christoffel symbols of the Levi-Civita connection from General Relativity. This makes sense since these coefficients in a sense act as the components of our general connection in our chosen coordinate chart. It is also clear in this context that these components of our local gauge potential do not transform as a tensor and thus are frame dependent quantities.

Next, we would like to formulate an analog of the exterior derivative in the context of a general gauge theory. To construct this in the most general way, we will once again have to return to the total space, P, of our G-bundle. In this space, recall that we constructed our usual tangent bundle, TP, and then we added the extra structure of an Ehresmann connection to our G-bundle. This induced a decomposition of our tangent bundle into the vertical bundle and the horizontal bundle, $TP = VP \oplus HP$. So, we were able to decompose any vector field on P into the sum of a horizontal vector field and a vertical vector field. Just as before, if V is a vector field on P, let V^h denote the horizontal component of the vector field. Then for a Lie algebra-valued k-form (or more generally a vector-valued k-form), $\alpha \in \Omega^k(P) \otimes \mathfrak{g}$, on P, we define the gauge covariant derivative, $D\alpha \in \Omega^{k+1}(P) \otimes \mathfrak{g}$ to be

$$D\alpha(X_1, \ldots, X_{k+1}) = d\alpha(X_1^h, \ldots, X_{k+1}^h) \tag{11}$$

where d is the exterior derivative on P which acts on the k-form part of α. In particular, if ω is our connection one-form on P, then we call the quantity $\Omega = D\omega$ the curvature two form. Now, just as was done with the connection one-form, we can find a local representation of Ω on the base manifold by performing the pullback of this tensor via a local section, s. So, we define

$$\mathcal{F} = s^*(\Omega) \tag{12}$$

and this quantity is called the local field strength. If we have two sections s and s' defined on the same neighborhood U of the base space, then the field strength transformation is given by

$$\mathcal{F}' = \mathrm{Ad}_{g^{-1}}\mathcal{F} \tag{13}$$

where $s'(u) = s(u)\cdot g(u)$. So, the local field strength transforms under the adjoint representation which makes the resulting pullback tensorial and if the gauge group is abelian, then the field strength tensor is independent of the choice of local section. There exists a very useful equation that relates the curvature two form to the connection one-form. This equation is called Cartan's structure equation, and it states that if X, Y are two vector fields over P, then

$$D\omega(X,Y) = d\omega(X,Y) + \frac{1}{2}[\omega(X), \omega(Y)] \tag{14}$$

or stated more concisely

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{15}$$

If we pull both sides of this equation back to the base space by a local section, s, and let $\mathcal{A} = s^*(\omega)$ we find that

$$\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]. \tag{16}$$

Electromagnetism as a $U(1)$ Gauge Theory

In this section, We will place the framework discussed thus far into the context of electromagnetism. While I will analyze the gauge group $U(1)$ in this paragraph it is worth noting that nothing is preventing me from discussing the gauge groups $SU(2)$ for the weak force or $SU(3)$ for the strong force. So, the theory extends to these other areas as well which is good for understanding the gauge theory of the entire Standard Model. Lets now take this entirely mathematical buildup and apply it to the case of a $U(1)$ gauge group and a principal $U(1)$-bundle with four dimensional space-time as the base space of the theory. Note that $U(1)$ is a one-dimensional abelian Lie group which can be realized as the unit circle in the complex plane. So, it can be parametrized by $e^{i\theta}$ for $\theta \in [0, 2\pi)$. Since $U(1)$ is abelian and one dimensional, it's Lie algebra, $\mathfrak{u}(1)$, is too. Here, the term abelian now applies to the commutator bracket. Note that while $\mathfrak{u}(1)$ is isomorphic to $\mathbb{R}$ as a Lie algebra, it is often useful to identify this space with $i\mathbb{R}$ so that the exponential function agrees with the usual Euler parametrization of the unit circle.

Now, pick a neighborhood U of some point in spacetime. let $-ieT$ be the sole generator of our $\mathfrak{u}(1)$ Lie algebra. Here we will group the gauge coupling, $-e$, in with the generator to distinguish it. e arises from the choice of Killing form on the Lie algebra, or in other words, a choice of isomorphism between $\mathfrak{u}(1)$ and $i\mathbb{R}$. The negative sign is necessary to make the Killing form positive-definite. T is usually taken to be 1, but I will keep it around for a bit to emphasize the Lie algebra component of objects. In order to generate a local gauge potential over our spacetime region, U, we have to pick a local section $\sigma_1 : U \to P$. Then for $x \in U$, $\sigma_1(x) = e^{-ieT\theta_1(x)}$ for some smooth real function $\theta_1 : U \to \mathbb{R}$ on spacetime. So, a choice of θ_1 ultimately determines a choice of gauge in the physics sense of the word. This means that the space of possible gauges forms an infinite dimensional vector space over the real numbers. Lets now see what a gauge transformation looks like in our theory. Let $\theta_2 : U \to \mathbb{R}$ be another smooth real valued function such that $\sigma_2(x) = e^{-ieT\theta_2(x)}$, let $\Lambda(x) = \theta_2(x) - \theta_1(x)$, and let $g(x) = e^{-ieT\Lambda(x)}$. Then,

$$g(x)\sigma_1(x) = e^{-ieT\Lambda(x) - ieT\theta_1(x)} = e^{-ieT\theta_2(x)} = \sigma_2(x). \tag{17}$$

Thus, $g : U \to U(1)$ is our transition function. Let ω be a connection one-form on our spacetime $U(1)$-bundle. Let $\omega = \tilde{\omega} \otimes T$, for some differential form, $\tilde{\omega}$, over the total space. Then $\mathcal{A}_i = \sigma_i^*(\tilde{\omega}) \otimes T$ are the local gauge potential with respect to each section. Since $U(1)$ is abelian, we have that $\mathrm{Ad}_{g^{-1}}(\mathcal{A}_i) = \mathcal{A}_i$. So,

$$\mathcal{A}_2 = \mathcal{A}_1 + e^{ieT\Lambda} d(e^{-ieT\Lambda}) = \mathcal{A}_1 - ieT \otimes d\Lambda \tag{18}$$

which is identical to the gauge transformation for the electromagnetic potential four vector given our identification of the $U(1)$ Lie algebra with $i\mathbb{R}$.

We can also construct the local field strength from this local gauge potential by applying Cartan's structure equation. Again, since $U(1)$ is abelian, the term $[A, A] = 0$. So, we are left with

$$F = dA \tag{19}$$

which is exactly the Maxwell field strength tensor. Since the gradings of the exterior algebra form an exact sequence with the exterior derivative acting as a connecting homomorphism, $d^2 = 0$. Thus,

$$dF = 0 \tag{20}$$

which yields all of Maxwell's equations. Thus, the assumption of this $U(1)$ gauge theory immediately derives Maxwell's equations. More generally, one can show that $D\Omega = 0$ for the curvature form in a general principal bundle. This result is called the second Bianchi identity. We also can see that this local field strength is invariant under gauge transformations since for an abelian gauge theory,

$$F' = F \tag{21}$$

for any gauge transformed local field strength, F'. In theories that postulate new physics beyond the standard model with additional $U(1)$ gauge groups, this leads to possible kinetic mixing terms when constructing a Lagrangian.

You will notice that I have not given a precise characterization of the Ehresmann connection, ω, on the $U(1)$-bundle, but instead I have elected to keep it general. Clearly though, we can perform experiments in the universe and produce values for the electric and magnetic field strengths which compose the F tensor. So, it must follow that what we measure in nature must correspond to some specific set of connection on our bundle and not just any arbitrary connection. To find out which connections it is, we will need to introduce a metric tensor on spacetime and utilize Hodge theory to construct the free field Lagrangian density,

$$\mathcal{L} = ||F||^2 = -\frac{1}{4} F \wedge *F. \tag{22}$$

The factor of $-\frac{1}{4}$ simplifies expressions deduced using this Lagrangian in physics and keeps results positive. This form of the Lagrangian is called the Yang-Mills Lagrangian. Recall that F is actually Lie-algebra-valued. So, we have to apply the killing form to the Lie algebra generators. Since we already grouped the gauge coupling with the generator, the Killing form on the generators, T, will satisfy $(T, T) = 1$. In our definition of $\mathcal{L}$, we implicitly take the inner product of the Lie algebra parts of F with itself so that $F \wedge *F$ is the familiar scalar quantity $F_{\mu\nu}F^{\mu\nu}$. The use of Hodge theory introduces another instance where an arbitrary positive coupling can be chosen, but for now we won't concern ourselves with it. Also, note that the combination $A \wedge *A$ is Lorentz invariant, but it is not gauge invariant. So, it is not written as an invariant term in our free field Lagrangian density.

The Euler-Lagrange equations can be shown to minimize the norm of F. So, the Ehresmann connection that gives us electromagnetism is the one that minimizes the size of the curvature tensor on spacetime. So, why couldn't we just demand that the curvature be zero? It turns out this is not possible for a $U(1)$-bundle and it is shown through Chern-Weil Theory that the fundamental properties of the "shape" of spacetime prevent the existence of Ehresmann connections with everywhere vanishing curvature.

It is worth noting that the equation of motion that comes from this Lagrangian is

$$\Delta F = 0. \tag{23}$$

Where $\Delta = D\delta + \delta D$ is the gauge covariant Hodge Laplacian operator and $\delta = *D*$ is the gauge covariant codifferential. In flat spacetime, $\Delta = \Box$. To satisfy this equation of motion, F must be a harmonic Lie algebra-valued differential form. It can be shown that the solutions of this equation are plane waves for the electric and magnetic field on spacetime.

The Lagrangian density can also be written in terms of a local gauge potential. Then the equations of motion yield

$$\Delta(dA) = d(\Delta A) = 0. \tag{24}$$

So, ΔA is a closed form and a simple solution to this equation is to pick $\Delta A = 0$. This is the Lorenz gauge choice in physics. If we think of A as our photon field, then $\Delta A = 0$ is exactly the massless Klein-Gordon equation for each component of the field. Thus, we are able to infer from this that the quantization of our gauge field must yield a massless particle (at the classical level) and must therefore propagate at the speed of light.

We also know that the A field transforms as a four-vector under Lorentz transformations since it is an element of $\Omega^1(M)$ which is identified via an appropriate solder form with an associated vector bundle of the Lorentz group (the reduction of the $GL(4,\mathbb{R})$-frame-bundle to an $SO(3,1)$-bundle is equivalent to the choice of a metric tensor on M). So, the field must carry the $(1/2, 1/2)$ irreducible representation of the Lorentz group. Thus, its intrinsic spin, s, must be equal to 1. So, we have found that the photon is a massless spin-1 particle just by assuming a $U(1)$ gauge theory.

I'll briefly mention how a source term, J, on spacetime can be implemented. In this case, we have the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} dA \wedge *dA - A \wedge *J. \tag{25}$$

Euler-Lagrange equations then yield the equation of motion, $\delta F = J$. This is the Yang-Mills equation for a classical source and it relates the presence of sources to the curvature of the $U(1)$-connection resulting from an action principal.

As an example of how gauge theory has impacted experimental physics, lets take a look at the Aharonov-Bohm effect. Lets consider a local region of spacetime that has a subregion of nonzero local field strength, however suppose that it is completely surrounded by a region of zero field strength. Now, consider a loop, γ, in spacetime that lies entirely in this region of zero field strength, but encloses the subregion of nonzero field strength. If we pick a base point for the loop and choose some initial element of $U(1)$ to associate to this point, say 1, then there exists a unique horizontal path lift of our loop. However, this need not be a loop in the total space of our fiber bundle. This means that the initial point may be different from the final point which we will call $g = e^{i\theta}$. To find g, we will consider an analog to the Wilson loop and let A be our local gauge potential in our region of spacetime. Then

$$g = \exp\left(\int_\gamma A\right). \tag{26}$$

We chose γ to be a loop that bounded some region, say Σ, in which the local field strength was non-zero. Now, we can apply Stokes theorem to find that

$$\int_\gamma A = \int_\Sigma dA = \int_\Sigma F. \tag{27}$$

Note that the result of these integrals is an element of the Lie algebra of $U(1)$. So, the exponential map results in an element of $U(1)$. We already claimed that the local field strength, F, was non-zero on Σ. So, there will be an overall non-zero phase,

$$\exp\left(\int_\Sigma F\right), \tag{28}$$

picked up by some particle if it traverses the loop. This Aharonov-Bohm effect is an interesting phenomenon whose solution feels natural in the setting of differential geometry and classical gauge theory.

It is also interesting to notice how the standard model predicts one gauge boson for electromagnetism and how the gauge group $U(1)$ is also one-dimensional. In fact, these two objects are intimately related by mixing in some quantum field theory to our model. Here, the local gauge potential is viewed as the photon field which, when quantized, has excitations that yield the familiar gauge boson of the standard model. The procedure constructed here is actually a general way of constructing all of the gauge bosons of the standard model. To see this, note that the dimension of $SU(2)$ is three. So, we have the three weak gauge bosons. For $SU(3)$, the dimension is eight. So, we have the eight different gluons of the standard model.

We can use the model of electromagnetism as a $U(1)$ gauge theory to provide reasoning for electric charge quantization. To do this, we need to briefly look at the irreducible representations of the gauge group. Since $U(1)$ is an abelian group, all irreducible complex representations will be one-dimensional. Note that $\mathbb{C}$ is a normed vector space (a special instance of a more general Banach space) with the norm defined by $|z|^2 = z^*z$ which contains the involution $z \to z^*$. This norm is only defined up to an overall positive number which introduces the possibility for another coupling. I find it most useful to think of $\mathbb{C}$ and the complex conjugate, $\overline{\mathbb{C}}$, as two separate spaces that are linked by the conjugation map. This norm allows us to look for unitary representations. So, we are looking for group homomorphisms $\rho : U(1) \to U(1)$. It turns out, these group homomorphisms are characterized by integers. Thus, each non-isomorphic unitary irreducible representation of our gauge group is labeled by a nonnegative integer, n, and

$$\rho_n(e^{-ieT\theta}) = e^{-ien\theta}. \tag{29}$$

Here, I am implicitly taking the Lie algebra generator acting on the representation space, $\mathbb{C}$, to be 1 for convenience. For a representation of $U(1)$ on $\mathbb{C}$, we can also construct the conjugate representation, $\overline{\rho}$, on $\overline{\mathbb{C}}$ by using the conjugation map. So, the representation of $U(1)$ on the conjugate space is given by the negative of the integer which is really the same representation as the representation characterized by the positive integer.

The differential of our irreducible representation, $d\rho_n$, is a representation of the Lie algebra $\mathfrak{u}(1)$, which when acting on the generator, is given by

$$d\rho_n(-ieT) = -ien. \tag{30}$$

This operator is skew-adjoint. So, lets divide the overall representation by i to get a self-adjoint operator on $\mathbb{C}$. Define

$$Q = d\rho_n(-ieT)/i \tag{31}$$

which we will call the charge operator.

Let $\psi : P \to \mathbb{C}$ be field which takes values in a $U(1)$ representation space, $\mathbb{C}$. If we want to construct covariant derivatives, then it is best to view ψ as a $\mathbb{C}$-valued differential 0-form on P, so that $D\psi$ is a $\mathbb{C}$-valued differential one-form on P. Since both P and $\mathbb{C}$ transform under an action of $U(1)$ we will require the field transform equivariantly under the two actions. This is essentially a compatibility requirement. For any $g \in U(1)$, ψ transforms as

$$\psi(p \cdot g^{-1}) = \rho_n(g) \cdot \psi = e^{-in\chi}\psi \tag{32}$$

for some $\chi \in \mathbb{R}$. Also,

$$Q\psi = \frac{1}{i} d\rho_n(-ieT)\psi = \frac{1}{i}(-ine)\psi = (-ne)\psi. \tag{33}$$

So, ψ is an eigenfunction of the charge operator with eigenvalue $-ne$ which we will interpret as the charge of the complex scalar field. Note that the representation theory of $U(1)$ quantizes the charges as integer multiples of the coupling constant e, but it makes no reference to what the fundamental unit of charge should be! This is why sometimes the generator of the $\mathfrak{u}(1)$ Lie algebra is given by $-ie/3$ since the most fundamental charge unit that has been discovered corresponds to the down quark charge magnitude of $e/3$.

We can also construct the complex conjugate field $\overline{\psi}$ in the conjugate space $\overline{\mathbb{C}}$ and we construct identically $Q^* = \frac{1}{i} d\rho_{-n}(-ieT)$. Then

$$Q^*\overline{\psi} = (ne)\overline{\psi} \tag{34}$$

So, $\overline{\psi}$ is a complex scalar field with charge ne. Thus, we see that complex conjugation of the irreducible unitary representations of $U(1)$ is equivalent to charge conjugation of fields. When quantized, this fact gives rise to antiparticles. Given this discussion, we can see that real scalar fields are uncharged.

Since we have an Ehresmann connection, $\omega \in \Omega^1(M) \otimes \mathfrak{u}(1)$ on our principal $U(1)$-bundle over spacetime, we would like to induce a connection on our associated vector bundle. This can be accomplished through the unitary representation map $\rho : U(1) \to \mathrm{GL}(\mathbb{C})$ whose differential is $d\rho : \mathfrak{u}(1) \to \mathfrak{gl}(\mathbb{C})$. We can use ρ to push forward the $\mathfrak{u}(1)$ piece of ω to $\mathfrak{gl}(\mathbb{C})$. Since, $\mathfrak{u}(1)$ is a one-dimensional Lie algebra, we can write $\omega = -\eta \otimes ieT$ for some one-form η. Then

$$d\rho_n(\omega) = \eta \otimes (-ien) = -ien(\eta \otimes 1) \tag{35}$$

We can then ask for the covariant derivative associated with this induced connection. Lets consider how it would act on the complex scalar field, ψ, over the total space, P. Applying our definition of the covariant derivative and letting X be a vector field over P shows,

$$D\psi(X) = d\psi(X^h) = d\psi(X + ien\eta(X)\sigma(1)) = d\psi(X) + ien\eta(X)d\psi(\sigma(1)). \tag{36}$$

Note that $d\psi(\sigma(1)) = -\psi$. So,

$$D\psi(X) = d\psi(X) - i(ne)\eta(X)\psi. \tag{37}$$

Hence,

$$D\psi = d\psi - i(ne)\eta\psi. \tag{38}$$

Then for some local section over spacetime, $s : U \to P$, $s^*(D) = d - i(ne)\mathcal{A}$ if $\mathcal{A} = s^*(\eta)$ and here $s^*(\psi) = \tilde{\psi} : U \to \mathbb{C}$ which lines up more closely with the definition of a complex scalar field on M. It is actually a theorem that the space of smooth equivariant maps, $\psi : P \to \mathbb{C}$ is isomorphic to the space of smooth sections of the bundle $\tilde{\psi} : M \to \mathbb{C}$, so we lost no information by considering our complex scalar fields to be equivariant maps on the total space instead of the usual interpretation of scalar fields with their domain as the base manifold.

There is a way to construct the bundle over M with fiber $\mathbb{C}$ which carries the $U(1)$ representation immediately from the principal bundle, P, but the construction is a bit technical, so I will neglect this for now and just comment that a complex vector bundle can be constructed on spacetime that carries an irreducible representation of $U(1)$. This is called an associated vector bundle and it is used to construct the charged matter fields, $\tilde{\psi}$.

The Hodge star acts in the usual manner on $\Omega^k(M)$ and if $\alpha \in \Omega^k(M) \otimes \mathbb{C}$ is a generic form, we can use the norm on $\mathbb{C}$ to contract the vector space components of α. If ϵ is a volume form on M, and (x, U) is a chart on spacetime, then $\alpha = \alpha_i dx^i \otimes z$ and

$$\alpha \wedge *\alpha = g_{ij} \alpha^i \alpha^j (z^* z) \epsilon \tag{39}$$

So, one of the fields will act like it takes values in the conjugate representation in the Lagrangian. Thus, demanding that the action be real implies that charged fields and their charge conjugates ought to both be present in the theory. Using this, the inner product on $\mathbb{C}$-valued differential forms which is defined by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta \tag{40}$$

allows us to construct real scalar Lorentz and gauge invariant combinations of our fields. So, for our complex scalar field, $\tilde{\psi}$, the most general action we can form by using the inner product on differential forms and complex norm on $\mathbb{C}$ is given by

$$\mathcal{S} = -\frac{1}{4}\langle F, F \rangle + \langle D\tilde{\psi}, D\tilde{\psi} \rangle - m^2 \langle \tilde{\psi}, \tilde{\psi} \rangle \tag{41}$$

where m is a possible mass coupling for the complex scalar field and note that this term is both Lorentz and gauge invariant, so it is allowed in the action. Thus, we have derived the charged complex massive scalar field Lagrangian that is familiar from classical field theory. From this, we see that Lorentz and gauge invariance of the Lagrangian is built-in and by Noether's theorem it is possible to derive the each of the important conserved quantities implied by this action such as charge conservation (from a global $U(1)$ symmetry in the Lagrangian). Similarly, the stress-energy tensor and the Hamiltonian can be constructed which are foundational to the canonical quantization of the theory.

Conclusion

Gauge theory provides a powerful framework for understanding the nature of our universe from a purely geometric point of view which (for me at least) is the most rewarding perspective. This framework also opens the door to other modern theories such as Yang-Mills theories, non-linear Sigma Models, and Chern-Simmons models. All of these are major areas of modern research. It is also interesting to see just how far classical field theory can reach into modern physics before quantizing the fields becomes important to further our understanding. In fact, while this procedure of quantization has been extremely fruitful in understanding modern physics, it has also given mathematicians headaches for some time to try to make these quantum field theories mathematically rigorous. For me though, this leaves classical gauge theory as one of the sturdy building blocks to start from in order to hopefully one day bring some formulation of quantum field theory back under the veil of mathematics.

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