Reductions in Principal Fiber Bundles and Gravitation

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Abstract

General Relativity, proposed by Albert Einstein in 1915, has been the cornerstone of gravitational physics for over 70 years, utilizing differential geometry. Despite advancements in mathematics, its core assumptions, particularly the equivalence principle, remain unchanged. This principle equates gravitation to accelerated reference frames, reducing spacetime's general coordinate invariance to Lorentz invariance via the metric tensor. This paper explores this perspective using the principal fiber bundle formulation, emphasizing geometric clarity, coordinate independence, and connections to classical gauge theory. We interpret the equivalence principle topologically, construct the frame bundle, and demonstrate how its reduction to a Lorentz group bundle yields the metric tensor, and draw comparisons with general relativity's framework.

1 Introduction

General Relativity has served as the keystone of gravitational physics for the past 70 years. This theory was first proposed by Albert Einstein in 1915 and is constructed using the language of differential geometry. While differential geometry as a field has seen considerable growth since the early days of Riemann, Gauss, and Cartan, the foundations of general relativity have remained for the most part unchanged since the days of Einstein. Nevertheless, the advancement of mathematics has allowed for a deeper understanding of the fundamental assumptions built into general relativity. The most fundamental assumption is the equivalence principle which states that gravitation is indistinguishable from accelerated reference frames. Broadly speaking, this statement is equivalent to the statement that our spacetime, which is invariant under general coordinate transformations, is reducible to a theory invariant under Lorentz transformations. The expense of this reduction is the metric tensor. From this viewpoint, the metric tensor plays the role of a classical Higgs field. While this perspective is not a common one taken in the literature, it paints an interesting geometric picture that is worth exploring.

As with any area of physics, there are a number of formulations of a theory which ultimately lead to the same description of nature. However, certain formulations have proved more enlightening and open to generalization. This is certainly true of general relativity. Throughout this paper, we will stick to the principal fiber bundle formulation of the subject for a few reasons:

1. It puts the physical geometry of the problem at center stage as opposed to other formulations where differential equations or fields on a fixed background manifold take the spotlight.

2. This formulation is independent of coordinate system. Thus, the "debauch of indices" as Michael Spivak called it ceases to cloud often times simple equations.

3. All of the methods used in this formulation carry over smoothly to the study of classical gauge theory. Thus, this one classical approach proves vital to other, seemingly disconnected, areas of fundamental physics.

It is also important to note that this paper is not intended to be a review of principal fiber bundle mathematics and modern geometry. For a review of this subject see ref. [5] So, this paper will try to keep some of the more abstract objects constrained to a general discussion and "sketch" certain aspects of the theory when needed.

2 Atlases, Charts, and Transition Functions

With the formulation fixed, we will view the equivalence principle as being topological at heart. To elaborate on this, recall that in modern geometry, real smooth n-dimensional manifolds are defined in a way which is independent of an embedding into an ambient space. They are defined as a topological space with a collection of neighborhoods and associated diffeomorphisms onto open subsets of $\mathbb{R}^n$ such that these neighborhoods provide an open cover of the manifold and the diffeomorphisms are compatible on overlapping neighborhoods. A maximal collection of such neighborhoods and associated diffeomorphisms to open subsets of $\mathbb{R}^n$ is called a smooth atlas for the manifold. A smooth atlas contains ample topological and differential information about a smooth manifold. In some cases, (Exotic $\mathbb{R}^4$ for example) there are many different choices for differentiable structures on the same underlying manifold. The combination of an open neighborhood and its associated diffeomorphism is called a chart and it is usually denoted by (x, U) where $x : U \to \mathbb{R}^n$ is a diffeomorphism. If we pick two such elements of the atlas, (x, U) and (y, V) such that $U \cap V \neq \emptyset$, then the map

$$\tau_{xy} = x \circ y^{-1} : y(U \cap V) \to x(U \cap V) \tag{1}$$

is called the transition function. We define the tangent bundle to be the collection of all points on the manifold and the tangent space associated to each point. We denote this bundle by TM and at each point $p \in M$, the tangent space at p is denoted by T_pM which carries the structure of a vector space over the real numbers. Note that by the compatibility requirement, the transition functions are diffeomorphisms. So, by the inverse function theorem, the differential of a transition function is an isomorphism of the tangent spaces

$$(d\tau_{xy})(p) : T_{y(p)}\mathbb{R}^n \to T_{x(p)}\mathbb{R}^n \tag{2}$$

for each $p \in U \cap V$. This map leads to the association of a group element $\tilde{\tau}_{xy}(p) \in GL(n, \mathbb{R})$ which transforms from one coordinate representation of the tangent space T_pM to the other. We can do this for all transition functions in the atlas. In this paper, it will be most useful to view the transition functions as maps

$$\tilde{\tau}_{xy} : U \cap V \to GL(n, \mathbb{R}) \tag{3}$$

where the overlapping charts are (x, U) and (y, V).

The content of the equivalence principle is that we may choose an atlas on a 4-dimensional manifold such that the transition functions take values in the Lorentz (actually spin) group which in this context is viewed as a subgroup of the full general linear group. These atlases cannot in general be found on manifolds and there exists a certain "topological obstruction" to being able to do this. Interestingly, it can be shown that 4-dimensional manifolds with zero Euler characteristic admit this kind of atlas.

3 The Frame Bundle and Group Actions

In the above discussion, we thought of transition functions as not only changing the coordinate representation of points on the manifold, but also acting as a linear transformation on the tangent spaces associated with those points. This idea of groups acting as transformations on certain spaces is pervasive in both mathematics and physics for the reason that they form an "ideal pairing" between algebraic objects. This means that it is possible to study the properties of each object through its interaction with the other object. The most useful type of action is when the group acts as a symmetry of the space. Since the above geometry has already led us to the action of the general linear group on the tangent vector space, we want to construct from this action a space in which the general linear group acts as the group of symmetries. A natural choice is the space of bases for the tangent space. This space is nice for the following reasons:

1. Each element of the group $GL(n, \mathbb{R})$ acts as a diffeomorphism of this space (to see this note that $GL(n, \mathbb{R})$ is diffeomorphic to the space of bases for the n-dimensional vector space).

2. For any two bases there exists a unique element of the general linear group which maps one basis onto the other (called a transitive action).

3. If any element of the general linear group fixes an entire basis, then this must be the identity transformation (called a free action).

This type of free and transitive action is precisely what is needed for the construction of a principal fiber bundle for $GL(n, \mathbb{R})$. In more detail, we define the base space of the bundle to be the n-dimensional manifold, M, and the fiber above any point $p \in M$, denoted F_p, to be the space of bases for the tangent space at p. Since all tangent spaces are vector spaces of the same dimension, the space of bases for that vector space are all isomorphic to one another and we will denote the generic fiber simply by F. The total space of the bundle is denoted FM and it can locally be described as the product space $U \times F$ for U an open neighborhood on M which is contained in a chart. Thus, charts are sometimes thought of as trivializing neighborhoods on the manifold. The general linear group acts on each of the fibers in the manner described above. This particular principal fiber bundle is called the frame bundle of M. Note that there is a natural smooth projection map, $\pi : FM \to M$ from the total space to the base space given by $\pi(p, e) = p$ where e represents any basis for T_pM. A local smooth right inverse for π is called a section, $s : U \to FU$ such that $\pi \circ s = \mathrm{id}_U$ for U an open subset of M. Thus, for an element $p \in U$, $s(p)$ is a choice of basis for T_pM which is called a frame. Since we have a smooth choice of these frames for each point of U, a smooth local section of the frame bundle is often called a moving frame in the literature. The space of all smooth local sections for an open neighborhood U is denoted $\Gamma(U, FU)$.

How does this space transform under the pointwise action of $GL(n, \mathbb{R})$? Consider a section $s \in \Gamma(U, FU)$. For each $m \in FU$, we can smoothly pick an element $g(m) \in GL(n, \mathbb{R})$. Then the combination $s(m) \cdot (g \circ s)(m)$ is again a smooth local section of the frame bundle. The space of all such smooth maps $g : FU \to GL(n, \mathbb{R})$ forms a group, $\mathcal{G}(U)$, which is infinite dimensional, and it acts freely and transitively on the space of smooth local sections of the frame bundle. Implicitly here, we require the map g to transform as

$$g(m \cdot h) \to h^{-1} g(m) h \tag{4}$$

for $h \in GL(n, \mathbb{R})$ in order to satisfy equivariance between structures. In the mathematics literature, the group $\mathcal{G}(U)$ is called the gauge group, while the group $GL(n, \mathbb{R})$ is called the structure group of the frame bundle. It is important to note that all automorphisms of the frame bundle which fix points on the manifold (so they only transform fibers into themselves) are of this form. These form a special class of bundle automorphisms called vertical automorphisms.

4 The Equivalence Principle

We want to apply the equivalence principle to the frame bundle on a four-dimensional manifold. This has been closely studied in the context of principal fiber bundle reductions in refs. [4, 3]. So, what happens to the frame bundle when we choose the atlas to have transition functions taking values in the Lorentz group? The Lorentz group does not act transitively on the space of bases for the tangent space. Instead, the space decomposes into orbits. For any fixed basis, e, in a generic frame, F, the orbit containing that basis is the subset

$$O_e = \{e \cdot h \mid h \in SO(3, 1)\} \tag{5}$$

where $SO(3, 1)$ is the Lorentz group and $\cdot$ represents the right action of the group on the basis. Note that the action of the Lorentz group on the elements of an orbit is both transitive and free. To say two elements belong to the same orbit is an equivalence relation on the space of frames. So, we can form the space of orbits which is denoted by $F/SO(3, 1)$. Note that this space is isomorphic to the coset space $GL(4, \mathbb{R})/SO(3, 1)$. Because the Lorentz group is a closed Lie subgroup of $GL(4, \mathbb{R})$, the coset space has the structure of a smooth manifold on which $GL(4, \mathbb{R})$ acts transitively (but not freely) and note that the dimension of this manifold is $16 - 6 = 10$. Now pick a global smooth section, $\sigma : M \to FM/SO(3, 1)$ of this quotient frame bundle. Explicitly, $\sigma(p) \in F_p/SO(3, 1)$ where $F_p/SO(3, 1)$ is the orbit space constructed by allowing the Lorentz group to act on the fiber lying above the point p. Thus, σ assigns to each point $p \in M$ a particular $SO(3, 1)$-orbit. In other words, σ identifies a "copy" of $SO(3, 1)$ immersed in a higher 10-dimensional space.

In fact, one can argue that the image of σ in this higher dimensional space is what is deserving of the name "spacetime", since the underlying four-dimensional manifold has no such characterization in terms of "space" and "time". In the mathematical physics literature, the map σ is referred to as a classical Higgs field. We can use the map σ to define a bundle reduction. While global sections of the frame bundle are not always possible (when they do exist the manifold is said to be parallelizable), the existence of a global section of the quotient bundle is guaranteed by the equivalence principle. Indeed, consider an atlas for spacetime such that all transition functions take values in the Lorentz group. In the quotient bundle, these transition functions act trivially. This implies that the quotient bundle is globally a product manifold which is equivalent to the existence of a global section of this bundle.

Let FM^{σ} denote the fiber bundle whose base space is M and the fiber above each point $p \in M$ is the space of all frames lying in the orbit $\sigma(p)$. Each orbit is isomorphic to one another, and is acted upon both transitively and freely by the Lorentz group. Thus, we have a principal fiber bundle with structure group $SO(3,1)$ and the principal $GL(4,\mathbb{R})$ frame bundle is said to be reduced to the principal $SO(3,1)$ bundle FM^{σ}. While a global section of the quotient bundle cannot in general be lifted globally to a section of the frame bundle, this can be done locally. In this paper, we will frequently consider the space of all local sections, $s : U \to FU$ such that $s(p) \in \sigma(p)$ for all $p \in U$ where U is a trivializing neighborhood of M. In this way, we can move the discussion of σ from the quotient frame bundle (which is more difficult to characterize) to the frame bundle which is easier to understand. From the above construction, we can see that there is a one-to-one correspondence between global sections of the quotient bundle and reductions of the $GL(4,\mathbb{R})$-principal bundle to an $SO(3,1)$-principal bundle.

5 The Metric Tensor as a Classical Higgs Field

The goal of this section is to identify the smooth global section of the quotient bundle $\sigma : M \to FM/SO(3,1)$ with the metric tensor. Lets explore this global section of the quotient bundle in more detail. At each point $p \in M$, the map $\sigma(p)$ picks out a particular $SO(3,1)$ orbit in the frame bundle. So, σ determines a set of privileged (inertial) frames out of all possible frames at that point. It remains to show that this is equivalent to a unique bilinear structure on T_pM. To do so, fix a point $p \in M$ and pick one such privileged frame, e. This determines an isomorphism $\tilde{e} : T_pM \to \mathbb{R}^4$. Let an element $h \in SO(3,1)$ act on e so that $e \cdot h = e'$. We can view h as being the unique linear transformation on T_pM which carries basis e onto e'. This is equivalent to the vector representation of h on $\mathbb{R}^4$. For convenience, we can use $\tilde{e}$ to view h in a matrix representation as a linear transformation from $\mathbb{R}^4 \to \mathbb{R}^4$. Since $SO(3,1)$ is the largest subgroup of $GL(4,\mathbb{R})$ to preserve on $\sigma(p)$, we want to identify a unique minimal structure on $\mathbb{R}^4$ which is invariant under the $SO(3,1)$ action. One way to identify such a structure is to lift this $SO(3,1)$ representation on $\mathbb{R}^4$ to the tensor algebra, $T(\mathbb{R}^4)$ and then look for fixed points. Note that $SO(3,1)$ has no non-trivial fixed points in $T^1(\mathbb{R}^4)$, however it does have a set of non-trivial fixed points in $T^2(\mathbb{R}^4)$. It is straightforward to show that these fixed points are all scalar multiples of the tensor $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$, or, using the dual basis, we have

$$\eta^{ab} = \mathrm{diag}(-1,1,1,1) \tag{6}$$

which is well recognized as the Minkowski metric tensor.

All higher rank fixed points in $T(\mathbb{R}^4)$ are tensor products of elements from this set of fixed points. So, this set is the minimal set which generates all of them. If we require the basis vectors to have magnitude of absolute value equal to one, then the overall scalar multiple can be fixed to one or minus one which yields the two possible metric signatures. By using the isomorphism $\tilde{e} : T_pM \to \mathbb{R}^4$ and extending it to the tensor algebra by $\tilde{e}(a \otimes b) = \tilde{e}(a) \otimes \tilde{e}(b)$, we can pull this metric on $(\mathbb{R}^4)^* \otimes (\mathbb{R}^4)^*$ back to a metric $g(p) \in T_pM^* \otimes T_pM^*$. Thus, in this basis, at this one point in M,

$$g_e(p) = \tilde{u}^{-1}(\eta) \text{ and } g_{\mu\nu} = \eta_{\mu\nu}. \tag{7}$$

What if we extend this analysis by a local moving frame, $s : U \to FU$, such that $s(p) \in \sigma(p)$ for all $p \in U$? Note that s need not be holonomic, nor is it a unique lift of σ from the quotient bundle to the frame bundle. Then the above analysis holds at each point in U with respect to this moving frame, s. So, again, $g = \eta$ everywhere in U.

Now pick a coordinate chart (x, U) on M. For each point $p \in U$, we have the holonomic basis for T_pM given by $\overline{x}(p) = \left\{\frac{\partial}{\partial x^\mu}|_p\right\}$. In sec. 3, we saw that local sections of the frame bundle transform under vertical automorphisms of the bundle, each of which is defined by a map $\gamma : FU \to GL(4, \mathbb{R})$. We can pick a vertical automorphism γ_x such that

$$\overline{x}(p) = v \cdot \gamma_x(v) \tag{8}$$

for all $v \in F_p$. So, the map, γ_x, is really a vertical automorphism which we can associate just to the chart (x, U). Now, let s be a lift of σ in FU and note that all such lifts are acted upon by the reduced gauge group $\mathcal{G}(U)$ in the reduced bundle. Then $\overline{y}(p) = s(p) \cdot \gamma_x(s(p)$. Note that for $h \in \mathcal{G}(U)$ we can define $s(p) = s'(p) \cdot h(s'(p))$. Recall that $\gamma_x(s(p))$ transforms as

$$\gamma_x(s(p)) = \gamma_x(s'(p) \cdot h(s'(p))) = h(s'(p))^{-1}\gamma_x(s'(p))h(s'(p)). \tag{9}$$

So,

$$\overline{y}(p) = [s'(p) \cdot h(s'(p))] \cdot [h^{-1}(s'(p))\gamma_x(s'(p))h(s'(p))] = s'(p) \cdot \gamma_x(s'(p)). \tag{10}$$

Thus, γ_x is independent of the choice of lift, s, and is really just dependent on the map σ.

We can allow $\gamma_x(s(p))$ to act on the metric tensor by lifting the action of $GL(\mathbb{R}, 4)$ on the tangent space at each point to the space of rank 2 tensors and we again use the $\cdot$ symbol to denote this action. So, in the coordinate chart (x, U) the metric tensor is $g_x(p) = \eta(p) \cdot \gamma_x(s(p))$ for any such s. In coordinates, $g_{\mu\nu}(p) = e^a_\mu(p)e^b_\nu(p)\eta_{ab}$. Note that by construction, the Minkowski metric tensor is a fixed Lorentz invariant, so

$$g_x(p) = \eta(p) \cdot \gamma_x(s'(p) \cdot h(s'(p)) = [\eta(p) \cdot h^{-1}(s(p))] \cdot \gamma_x(s'(p)) = \eta(p) \cdot \gamma_x(s'(p)). \tag{11}$$

Thus, $g_x(p)$ is also independent of the choice of lift of σ. So, it is therefore a property of the reduced bundle expressed in the chart (x, U). Here, the Roman letters label the basis vectors of $\mathbb{R}^{1,3}$ while the Greek letters label the holonomic basis vectors associated to the chart (x, U).

In general relativity, the local field

$$\gamma_x(s(p)) : U \to GL(4, \mathbb{R}) \tag{12}$$

is called a tetrad field (in geometry, this is really just a section of a particular associated bundle called the adjoint bundle).

We can say that $g_{\mu\nu}$ is the coordinate representation of η by thinking about the previous paragraph where we associated privileged frames in FM to the metric tensor η at a point $p \in M$. In this line of thinking, we picked a privileged frame $e \in F_p$ and thought of it as an isomorphism to $\mathbb{R}^4$. This gave us a metric tensor on $\mathbb{R}^4$. To get the metric tensor on T_pM, we needed to pull back $\eta \in (\mathbb{R}^4)^* \otimes (\mathbb{R}^4)^*$ by the map $\tilde{e}$. What if we had chosen some other frame e' potentially not related to e by a Lorentz transformation? Then by choosing $h \in GL(4, \mathbb{R})$ we could relate e and e' by $e' = e \cdot h$. Here, the element h is interpreted as an isomorphism $T_h : [\mathbb{R}^4]_e \to [\mathbb{R}^4]_{e'}$ which carries basis e onto basis e'. Then the metric tensor $\eta = \eta_{ab}$ in basis e transforms according to

$$[T_h]^a_\mu [T_h]^b_\nu \eta_{ab} = g_{\mu\nu} \tag{13}$$

when expressed in basis e' on T_pM. These operations are analogous to what is done in the previous paragraph, however they are now interpreted in terms of coordinate representations.

6 Summary of the Construction

From this, we can see the moral of the story. By applying the equivalence principle, we can pick a global smooth section $\sigma : M \to FM/SO(3,1)$ of the quotient bundle to perform the bundle reduction. This yields a choice of inertial frames lying in the fiber of the frame bundle over any chosen point, p, on the manifold. Up to metric signature, we were able to identify a unique metric tensor on T_pM which identifies each basis in the set of inertial frames as being orthonormal. When expressed locally in terms of any moving frame, s, which is a lift of σ to the frame bundle, the metric has the form $g_{ab} = \eta_{ab}$. This can be related to any local holonomic frame (i.e. in a coordinate system) (x, U) by using a tetrad field, $\gamma_x \circ s$. When the tetrad field acts on the metric tensor, it provides the local coordinate representation of the metric tensor

$$g_x(p) = \eta(p) \cdot \gamma_x(s(p)) \tag{14}$$

which is determined solely by the choice of coordinates and the choice of bundle reduction. Thus, from more of a field theory perspective, the above analysis says that it is valid to fix a coordinate system and then vary over possible metric tensors $g_{\mu\nu}$ to understand the space of possible bundle reductions. Also, the bundle reduction breaks the symmetry group associated to the entire manifold from the diffeomorphism group down to the isometry group.

7 Connections and Metric Compatibility

Now that we understand how the reduction of the frame bundle gives rise to pseudo-Riemannian metrics on spacetime, we can think about what happens to the space of connections on spacetime when this bundle reduction is performed. Connections on principal fiber bundles are defined geometrically by Ehresmann connections. Without getting into too many technical details, these are differential one-forms defined on the total space of the principal bundle and which take values in the Lie algebra of the structure group. Geometrically, these connections define a particular subspace (called a horizontal subspace) of the tangent space to the total space of the principal bundle at each point of the total space. More concretely, if FM is the frame bundle over spacetime M and $\pi : FM \to M$ is the natural projection map, then at each point $e \in FM$, we can always define a particular subspace (called the vertical subspace at e of FM) by $V_eFM = \ker d\pi_e$ where $d\pi_e$ is the differential of the natural projection. This subspace consists of vectors which are tangent to the fiber at e when the fiber is viewed as a submanifold of the total space. There does not exist, however, a canonical choice of complementary subspace, H_eFM such that

$$T_eFM = V_eFM \oplus H_eFM \tag{15}$$

for each point $e \in FM$.The smooth choice of such a complementary (horizontal) subspace at each point of the total space is equivalent to a choice of Ehresmann connection on the bundle.

Before any bundle reduction, the structure group is $GL(4, \mathbb{R})$ with Lie algebra $\mathfrak{gl}(4, \mathbb{R})$ and after the bundle reduction the structure group is $SO(3, 1)$ with Lie algebra $\mathfrak{so}(3, 1)$. We identify $\mathfrak{so}(3, 1)$ as a Lie subalgebra of $\mathfrak{gl}(4, \mathbb{R})$. Note that, as a vector space, $\mathfrak{gl}(4, \mathbb{R})$ splits as

$$\mathfrak{gl}(4, \mathbb{R}) = \mathfrak{so}(3, 1) \oplus \mathfrak{m} \tag{16}$$

where $\mathfrak{m}$ is a 10-dimensional vector space (not a Lie algebra) which is invariant under the adjoint action of $SO(3, 1)$.

Let ω be a connection on the frame bundle FM and let FM^σ be the sub-bundle of FM defined by the global section of the quotient bundle, σ. A connection is said to be reducible to a connection ω' on FM^σ if and only if on the sub-bundle, FM^σ, ω takes its values in the Lie subalgebra, $\mathfrak{so}(3, 1)$. Thus, on the restriction of ω to FM^σ, $\omega = \omega'$ which is a valid connection on FM^σ. Now, σ picks out a smooth set of privileged frames in the frame bundle at each point and view this set as a submanifold of the frame bundle. This submanifold determined by σ has its own tangent bundle which we can view as a sub-bundle of the tangent bundle of the frame bundle. There exists a theorem [2] which states that ω is reducible if and only if σ is parallel with respect to ω. In other words, for any lift, s, of σ to the frame bundle, its differential

$$ds(p) : T_pM \to T_{s(p)}FM \tag{17}$$

has its image satisfying $ds(p) \subset H_{s(p)}FM$.

To understand how this translates to a condition on the metric tensor, we need to view the bundle $T^*M \otimes T^*M$ as an associated vector bundle to the frame bundle. The representation that facilitates this is $\rho \otimes \rho$ where ρ is the (dual) fundamental representation of $GL(4, \mathbb{R})$. Now apply the following theorem: The space of sections of the bundle $T^*M \otimes T^*M$ is isomorphic to the space of smooth equivariant maps [1],

$$\psi : FM \to T^*M \otimes T^*M. \tag{18}$$

The term equivariant means that

$$\psi(e \cdot h) = \psi(e) \cdot (\rho \otimes \rho)(h). \tag{19}$$

Since the metric tensor, g, is a global section of the bundle $T^*M \otimes T^*M$, there exists a unique global map

$$\psi_g : FM \to T^*M \otimes T^*M. \tag{20}$$

So, to put things together, we have the diagram shown in Fig. 1:

Thus, we see that locally, the metric tensor can be expressed as $g = \psi_g \circ s$ and notice that this expression looks eerily similar to the expression for the metric tensor in Eq. 14 using tetrad fields. Now, we have function composition with an equivariant map. By using ψ_g, we can push the horizontal sub-bundle HFM forward to a horizontal sub-bundle of $T(T^*M \otimes T^*M)$. We can then differentiate the metric tensor locally in some chart (x, U) on M to get $dg = d(\psi_g \circ s)$ and factor this map as

$$dg = d\psi_g \circ ds. \tag{21}$$

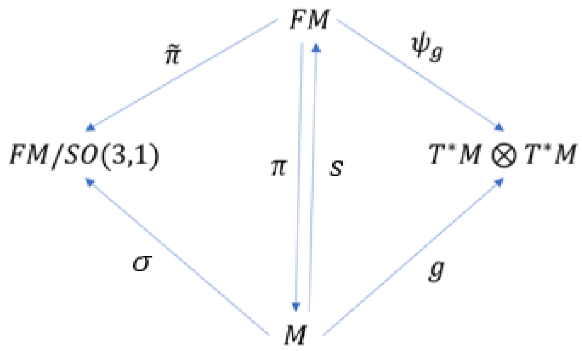

Figure 1: A visual diagram relating the metric tensor and the bundle reduction.

Recall that the differential of s takes values in the horizontal sub-bundle over the tangent bundle, so dg also takes values in the horizontal sub-bundle induced on $T^*M \otimes T^*M$. So, g must also be parallel with respect to the connection. This is essentially the statement of metric compatibility. So, we see that the only connections on the frame bundle which are preserved by the bundle reduction are those which are compatible with the metric tensor.

8 Conclusion

Principal fiber bundles on four dimensional.ConcurrentHashMap manifolds provide a unique, geometrically motivated, perspective on the construction of physical laws. In this paper, we first began with a very general construction of the frame bundle with structure group $GL(4, \mathbb{R})$. We then applied the equivalence principle and showed that this principle satisfies the requirements to reduce the $GL(4, \mathbb{R})$ principal bundle to a $SO(3, 1)$ principal bundle. Each way of possibly doing this reduction on the four-dimensional manifold is equivalent to the choice of a pseudo-Riemannian metric tensor on spacetime. By using our transformation laws for fields in each of the bundles, we saw how these fields transformed under the different gauge groups. Finally, we thought about what connections on the frame bundle are able to be reduced to connections on the $SO(3, 1)$ bundle and found that the allowed connections are exactly those which are compatible with the metric tensor.

Up to torsion, this provides the correct set of connections which are studied in general relativity. Moreover, it identifies the metric tensor as a classical Higgs field for breaking the general coordinate transformations down to Lorentz transformations. Thus, the remaining group structure which remains is the Lorentz group and all fundamental particles can then be constructed as sections of associated vector bundles over spacetime since they live in the various representation spaces for the Lorentz group. With field theory in mind, it is also worth pointing out that while this approach uses all of the tools associated with classical gauge theory, this is not a typical gauge theory for gravitation.

Additionally, the space of all field configurations is the same as the space of all bundle reductions which is the same as the space of all maps $\sigma : M \to FM/SO(3, 1)$. By picking a metric signature, all of the information in σ can be transferred into a metric. Then by picking the Levi-Civita connection on spacetime, we can transfer all of the information from the metric into the connection and this is the primary object studied in classical gauge theory. In this way, we can see how the information is just repackaged from the highly geometric formulation of the theory presented in this paper to a more standard interpretation as fields on a fixed background.

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