An Exploration of Higher Dimensional Objects

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Abstract

In order to develop a framework in which to gain a more intuitive understanding of the objects with more than three spacial dimensions, a ratio β was defined and computed for the three n-dimensional regular polytopes as well as for various radially symmetric objects. A closed-form expression for β was found for all objects that were studied. From this, it was found that the number of dimensions of a shape can be determined from only this β and its the general geometry. Therefore, it was possible to redefine a dimension in terms of a single quantity which depends only on two lengths. A possible relationship between β and the symmetries of various families of shapes was suggested and explored but has not been proven.

I. Introduction

When we first learn mathematics as children, we rely strongly on our intuitions about the world to understand and contextualize the formalism we are taught. We think about subtraction as how many apples we have left if we had 5 apples and gave 3 to our friend. When we do this, we can "see" the apples in our head. We are given blocks to represent units, tens and hundreds and we cut chocolate bars into thirds to get a visual (and tasty) understanding of fractions. As we continue in our mathematical education(at least at first glance), the concepts we are taught seem to become less and less intuitive and further divorced from our basic understanding of the world we live in.

One element of mathematics that often causes people to seperate their understanding of mathematics from their intuitions about the world is the concept of "higher dimensions" (more than three dimensions). The goal of this project is to bridge the gap between abstract mathematics and our intuitions about the physical world by studying objects with more than three spacial dimensions.

Note to the reader: This paper is meant to be accessible to those without much prior knowledge of mathematics and assumes only a basic knowledge of elementary geometry, a very basic familiarity with elementary calculus and some familiarity with basic trigonometric functions for all sections, except certain footnotes and the appendices, which are aimed at the more mathematically knowledgeable reader. For those concepts which I do not fully explain and are not central to the results presented here, I have provided interesting and accessible references when possible.

II. Motivation

How do we define a dimension? Naively, we might say a dimension is a direction one can travel that is perpendicular (or orthogonal) to all other directions in which one can travel. This seems to match our day to day understanding of the world. We can go in the right/left direction, the front/back direction and the up/down direction (although this last direction generally requirest he help of some technological contraption). This makes three directions, which agrees with our understanding that we live in a world with three spacial dimensions[1].

This definition, however, does not seem to help us intuitively understand what it means for an object to have 4 dimensions, much less 10 or 20. I can't point in a tenth direction any more than I can point in a fourth direction. Edwin A. Abbott highlights this well in his 1884 essay *Flatland* [Abbott, 1884], in which he introduces us to a square that lives in a 2D world (Flatland). The square cannot conceive of a third dimension; of what it means to go "up", even though we know that up does indeed exist. Like the square, we cannot conceive of a direction we cannot see, one that is not a part of our world. Therefore, we are left with the fact that either we cannot gain a profound understanding of these higher dimensional spaces that are so often studied in mathematics and physics or we must find another scheme by which to comprehend these concepts.

Another seemingly natural way to understand what a dimension is could be to consider a dimension as a piece of information needed to localize an event in the world. For example, in our universe, which has three spacial dimensions and one-time dimension, we need four data points to localize an event: a longitude, a latitude, an altitude and a time. While this is seemingly just a consequence of the previously discussed idea of directions, it also provides us with a possible path to explore in search of a better way to understand higher dimensional objects and spaces. A set of coordinates in space, like we just described, assumes the idea of an origin. From this, the concept of vectors, and therefore distances, arises. The work presented in this text is centred around the scaling of distances in spaces with an increasing number of dimensions.[2] This came from the logical progression of concepts above, but also from the fact that the distance between points is a somewhat fundamental and certainly simple and intuitive quantity to study in a geometric shape.

III. Methodology

Before starting our dive into the more quantitative portion of the work, I'd like to discuss some aspects of the methodology that was used and give some justification for the steps that were taken, starting with a short discussion of where the idea for this method came from. In physics, there is a famous series of experiments which we call the Stern-Gerlach experiments. The general idea is that we send particles into a black box and can measure them after they come out, which gives us information about them. While we don't necessarily know how this box works, we know that it does indeed work and makes no errors[3].

While we will not dive into the very profound but somewhat complex results and implications of this particular set up, I think it is important to understand its premise, both for context and justify the thought experiment I am proposing. There are a few important takeaways that I would like to touch upon before we leave the Stern-Gerlach experiments behind[4]. First, it is important to note that how we get the particles and where they come from or are prepared is irrelevant and doesn't affect the results or the information we can gain from the exercise. Furthermore, the internal

[1] I will only consider spacial dimensions for the purpose of this exercise. If you wish, you can imagine that the objects I will be considering live in a space which also has an extra temporal dimension, but that we are observing a snapshot of them in time.

[2] For simplicity, I have considered only objects embedded in spaces with Euclidean metrics, or so-called "flat" spaces. It would be an interesting, if more complex and computationally involved, extension to the project to check if this generalizes to other types of metrics and if, perhaps, this could provide us with some more simple or elegant intuition.

[3] Sometmimes, we imagine an "intelligent monkey" which lives in the box and is the cause of the effect the box has on the particles.

[4] The Stern-Gerlach experiments were performed in a lab, but for the purposes of our understanding, this is not important.

mechanism of the black box need not be known for the thought experiment version to give us insight.

Now, let me propose a similar game to play, the results of which I will be developing throughout this paper. Imagine a similar set up to the Stern-Gerlach experiments, but rather than particles entering the black box, they are shapes or abstract geometries. We don't know the number of dimensions of the shape but we would like to find out. The goal of this project is therefore to develop some idea of a quantity that the black box could give to the user which would give us the number of dimensions.

Those of you who remember some elementary geometry might suggest that volume could be this quantity and you would be correct in claiming that volume could fill this role. In up to three dimensions, this is fairly obvious. A line that is doubled in length would double in mass as well, if we assume constant mass density, whereas a 2D tile would quadruple in mass if its side length was doubled, since area scales as x^2. Similarly, a box would be eight times as heavy if its side length was doubled because 3-volume scales as x^3. While I won't offer any proof of this, nor will I dwell on this example because it will not be relevant to our future discussion, I will ask the reader to accept that n-volume scales as x^n, which means that we could conceivably, use volume in our thought experiment. However, volume in more than three dimensions is not very intuitive, so we will try to find a better quantity to examine.

IV. Regular Polytopes

i. Regular Polygons

Many of us were introduced to the concept of regular polygons at some point during our first or second journey into the wonderful world of geometry. We define them as having sides of equal length and equal interior angles. Because of this regularity and predictable progression, I began my exercise by considering these shapes.

A natural first question to ask, if we wish to be able to generalize any relation found or glean some understanding of higher dimensions from it, is whether we can find a more general or intrinsic way to define regular polygons. For this, we will take a ratio of two distances, in order to be able to make this measure independent of the length of the sides of the shape.

A natural choice of two distances would be the shortest straight line segment from the center of the shape to one of its exterior boundary (which turns out to be the segment joining the center of the shape to the center of a side) as well as the distance from the center of the shape to a vertex, as shown in Figure 3 . These will turn out to be good choices of distances, since they will be generalizable not only to all regular polygons but also to all regular shapes in an arbitrary N dimensions.

Notation: We will denote the shortest of the two distances d and the larger of the two D.

Definition 1. *We will call the ratio of interest β [5], which we define as $\beta := \frac{d}{D}$.*

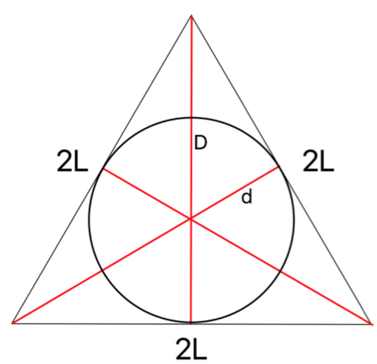

Figure 1: *Inscribed circle in an equilateral triangle.*

An important fact to note is that these distances d and D will correspond to the radii

[5] I have designated the ratio of interest as β in honour of the BLUE Fellowship, during which this research was completed (β being the Greek equivalent of B).

of the inscribed and circumscribed circles respectively.

Definition 2. *The **inscribed** cricle is the circle tangent to each side of the shape. See Figure 1.*

Definition 3. *The **circumscribed** cricle is the circle touching each vertex of the shape. See Figure 2.*

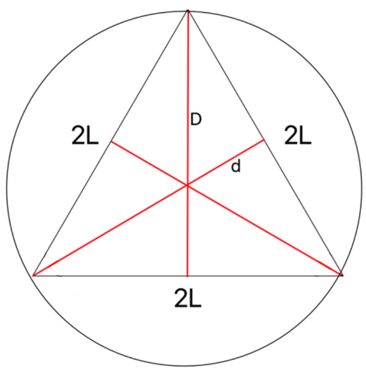

Figure 2: *Circumscribed circle around an equilateral triangle.*

First, let's compute β for the equilateral triangle. Given sides of length 2L, let's first calculate the length D. This is simply a question of simple geometry and the Pythagorean theorem. Since the goal here is not to provide a full proof but rather to build intuition, I will not provide a full proof but rather a short sketch of the method to compute this distance.

In Figure 3, we take the shaded triangle, which we know must be an isosceles triangle, with interior angles $\frac{\pi}{6}$ and $\frac{2\pi}{3}$. From this, we can create two right triangles Figure 4, of side lengths d, D and L.

By definition, we know:

$$\tan(\theta) = \frac{d}{L} \tag{1}$$

Using this and the Pythagorean Theorem, we can solve for β.

$$\beta = \frac{d}{D} = \frac{tan(\theta)}{\sec(\theta)} = \sin(\theta) \tag{2}$$

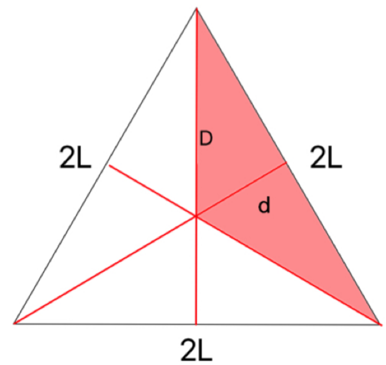

Figure 3: *An equilateral triangle with sides of length 2L with D and d labelled. The center of the shape is where the three red lines intersect.*

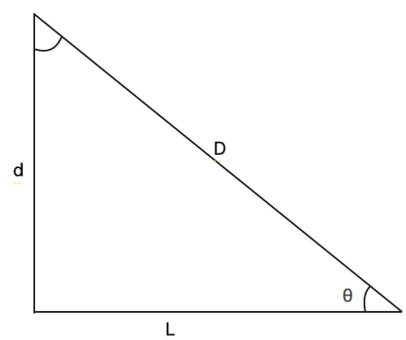

Figure 4: *A right triangle made of half the shaded area in Figure 3.*

We simplify this, knowing $\theta = \frac{\pi}{6}$ to get that for equilateral triangles:

$$\beta = \frac{d}{D} = \frac{1}{2} \tag{3}$$

Now, let's compute the ratio for a square of side length $2L$, shown in Figure 5.

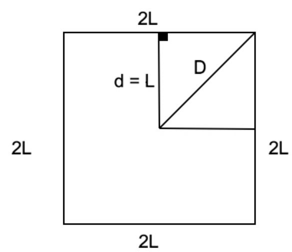

Figure 5: *A square, showing D and d.*

Again, the computation is quite simple and we get:

$$\beta = \frac{L}{\sqrt{L^2 + L^2}} = \frac{1}{\sqrt{2}} \tag{4}$$

Since we know that the angles of the right triangle are both $\frac{\pi}{4}$, we can see that this is equivalent to the result found in Equation 2, since $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

While I won't give a formal proof of this, it is easy to convince yourself that the relation found in Equation 2 applies to all regular polygones. Since the interior angle of a regular polygone with s sides is $\frac{(s-2)\pi}{s}$, we know that $\theta = \frac{(s-2)\pi}{2s}$. Since $s > 3$, we know that θ will stay bounded and will never be smaller than $\frac{\pi}{6}$. The upper limit of θ is:

$$\begin{aligned}\theta_{max} &= \lim_{s\to\infty} \frac{(ns-2)\pi}{2s} \\ &= \lim_{s\to\infty} \frac{\pi}{2}(1 - \frac{2}{s}) \\ &= \frac{\pi}{2}\end{aligned} \tag{5}$$

As a sanity check, let's consider what this limit is. If a regular polygon has an infinite number of sides, it can be considered a circle. Since a circle is defined as a shape where all points are equidistant from the center, β should be 1. Indeed, we see that the ratio will be $\sin\left(\frac{\pi}{2}\right) = 1$, which should not be a surprising result, but one we are glad to have obtained nonetheless. It is also important to note that since θ varies between $\frac{\pi}{6}$ and $\frac{\pi}{2}$, $\sin(\theta)$ will not behave periodically and will take a different value for all values of s. So, it will be different for every regular polygon. This gives us a semblance of uniqueness.

With this result in hand, we have shown that the ratio of the radii of the inscribed and circumscribed circles is a good identifier of a shape. However, this number and the evolution of this number is not particularly intuitive. In order to transform this into what I believe to be a slightly more intuitive form, we will look at the $s = 3$ and $s = 4$ cases computed above[6] and rewrite them as

$$\beta = \frac{1}{\sqrt[s-2]{n}}. \tag{6}$$

where n is the number of dimensions (here, 2). We will show that this is true in the next section. For $s = 3$, we get $\frac{1}{\sqrt[1]{2}} = \frac{1}{2}$. For the $s = 4$ case, we get $\frac{1}{\sqrt[2]{2}} = \frac{1}{\sqrt{2}}$. Both of these are as desired.

ii. Higher Dimensional Regular Polytopes

Polytopes are the higher dimensional analogues of polygons [Adams, 2013] [Mansiska, 2008]. For example, a cube is simply a $n = 3$ polytope. Similarly, a regular polytope is a polytope with identical boundaries and equal distance from the center of the polytope to each vertex. We know that there are 5 regular polytopes in three dimensions, which we call the platonic solids. These were first introduced and studied by Plato (which you may have guessed by their name), but we wouldn't consider polytopes in more than 3D until about 1850, through the work of Schläfli [Souam, 2004].

[6] These are the only shapes that will generalize to an arbitrary N dimensions, so they will be the only ones considered.

Before diving into our discussion of polytopes, we will take a moment to recall three of the five platonic solids. These will be useful to us in higher dimensions.

Definition 4. *The* ***cube*** *has 6 faces, each of which a square. It is the 3D extension of the square.*

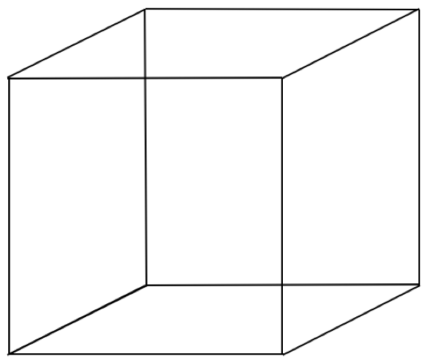

Figure 6: *The cube.*

Definition 5. *The* ***tetrahedron*** *has 4 faces, each of which is an equilateral triangle. It is the 3D extension of the equilateral triangle.*

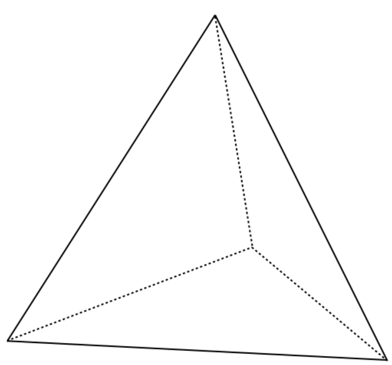

Figure 7: *The tetrahedron.*

Definition 6. *The* ***octahedron*** *has 8 faces, each of which is an equilateral triangle. It can be useful to imagine the shape as two pyramids with a square base glued together at their bases. It is not an obvious 3D extension of any regular polygon.*

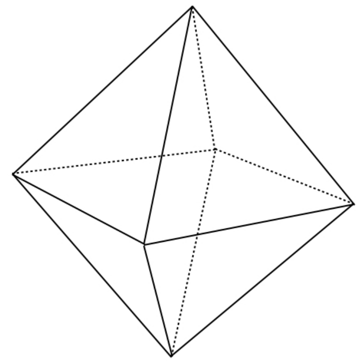

Figure 8: *The octahedron.*

We now know that there are a finite number of polytopes in every dimension higher than 2 [7]. In fact, in dimensions higher than 4, only three regular polytopes exist. These are the higher dimensional extensions of the cube, tetrahedron and octahedron.

In three dimensions, it is fairly straightforward to compute β for the cube. The distance D is, in fact, simply the distance between the origin and a point situated at (L, L, L), if the cube has sides of length 2L. The distance d is simply L like it was for the square. If you are having trouble picturing this, you can convince yourself by thinking of these distances in terms of vectors in 3-space, and imagine the cube as being centred at the origin. So, for the cube, we get:

$$\beta = \frac{L}{\sqrt{L^2 + L^2 + L^2}} = \frac{1}{\sqrt{3}} \tag{7}$$

Given what we know about distances in euclidean space[8], we can generalize this fairly easily. In fact, centering a the cube (of n dimensions) at the origin, βwill be:

$$\beta = \frac{L}{\sqrt{nL^2}} = \frac{1}{\sqrt{n}} \tag{8}$$

[7] If you'd like to know more about why this is or what they are, the Youtube channel Numberfile has a wonderful and very accessible video on the topic called "Perfect Shapes in Higher Dimensions".

[8] Euclidean space is simply flat space, or the space you are probably used to dealing with. It is the x-y-z system used in introductory geometry.

This result will give us a lot of valuable information, but I will put off the discussion of the implications of this statement as well as the intuition we can hope to get from it until after we have made the same computations for the other regular polytopes. I will, however, pause to note that this is consistent with the 2D case from the last section, as expected.

The computations for the tetrahedron and octahedron are significantly more involved, mostly due to the nature of tilted triangles. Therefore, I will use the results for the length of the radii of the inscribed and circumscribed n-spheres (the n-dimensional extension of the sphere) stated in *Radii of Regular Polytopes* [Brandenberg, 2003] and the fact that the center of a regular tetrahedron is $\frac{1}{3}$ of the way up its axis of symmetry. It will also be useful to know that this proportion scales as $\frac{1}{n}$ in higher dimensions.

However, knowing this last fact we have already found β. If this is not immediately obvious to you in terms of radii of circumscribed and inscribed spheres, you can convince yourself by thinking of these distances as shortest and longest distances from the center. Therefore, we get that for an n-simplex of side length 2L:

$$\beta = \frac{1}{n} \tag{9}$$

which is consistent with the 2D case, as desired.

Finally, we will look at the octohedron. We know, from [Wu & Zhang, 2010] that β is:

$$\beta = \frac{1}{\sqrt{n}} \tag{10}$$

This is interesting since this is the same relation as for the cube. There is a profound reason for this [9], but I will not discuss this in depth because it will shed little light on the immediate question at hand.

[9] If you are familiar with the concept of a dual space, you can check that this makes sense, knowing that the octahedron is the dual of the cube.

Before making this computation for other shapes, I will take a moment to discuss the implications of the relations we have just found.

iii. Brief Discussion of the Results of Section IV

The results obtained in this section have a pleasing symmetry to them. In fact, all of the regular polytopes follow a similar evolution, where β evolves as "one over some power of the dimension". This may lead us to think that our β is, in fact, a good quantity to think about. The facts that we can find a closed form solution for β in every dimension and that the evolution of β as a function of dimension is so regular, in addition to the aforementioned regularity of the closed form solution between the shapes all support our choice of quantity. With this in hand, we can compute β for more shapes to see if we can indeed build some type of intuition for higher dimensions.

V. Radially Symmetric Objects

The most symmetric and simplest shape we have is the sphere. However, β will always be 1, independent of dimension. For this reason, the sphere is not particularly interesting for our discussion so we will move on.

i. Tori

The torus is a commonly studied object. You can picture the torus as being the surface of a very smooth bagel, as seen in Figure 9.

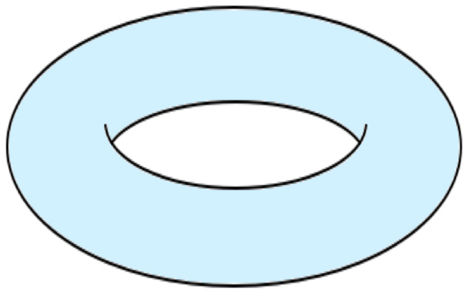

Figure 9: *The Torus.*

An interesting thing to stop and consider about the torus is that while it is a 2D object (it is, after all, a surface), it can only exist in a space that has more than two dimensions. One might be tempted to say that a torus embedded (living) in two dimensions is simply an oval inside another oval, but the two lines won't be connected which is why we cannot consider this a torus in a plane[10]. This difference, while fundamental, will not greatly affect the nature of our discussion. However, it will affect the notation, which we will need to keep in mind.

Notation: An **n-torus** will refer to a torus embedded in a space with n+1 dimensions. Therefore, an n-torus is, in fact, an n-dimensional object.

We define the the 2-torus as follows:

$$(\sqrt{x^2+y^2}-R)^2+z^2=r^2 \tag{11}$$

where R and r are as shown in Figure 10

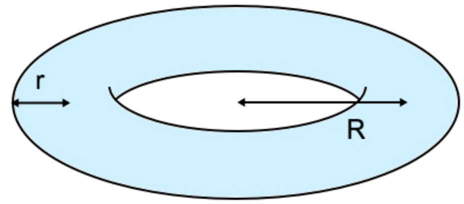

Figure 10: *R and r on the torus.*

We will parametrize[11] this torus in the usual way as follows:

$$\begin{aligned} x &= (R + r\cos\theta)\cos\varphi \\ y &= (R + r\cos\theta)\sin\varphi \\ z &= r\sin\theta \end{aligned} \tag{12}$$

where θ and φ are as shown in Figures 11 and 12:

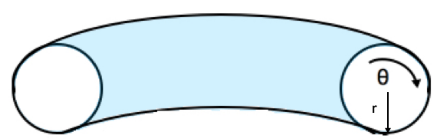

Figure 11: *θ on the torus.*

[10]This is a very informal argument, but a formal argument could be made using the concept of simply connected surfaces. Again, I won't take the time to discuss this in depth since the goal is simply to gain intuition.

[11]If you are not familiar with the concept of parametrization, you can imagine it as a change of variables. In simple terms, we want to write our coordinates x, y and z in terms of other variables such that equation (11) is still true.

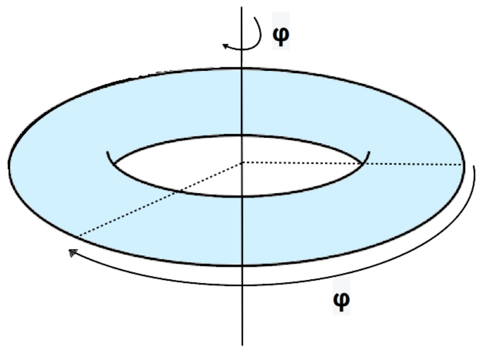

Figure 12: *φ on the torus.*

We could parametrize the 2-torus differently, but this turns out to be a convenient parametrization so we will keep it. If you are not convinced that this is an appropriate parametrization, it might be worthwhile to take a moment to substitute x, y and z for the expressions in (12). With a little bit of trigonometry, you should easily be able to see that equation (11) holds.

The choice of D and d on the torus is not quite as obvious as it was for the regular polytopes. What is the inscribed sphere? The circumscribed sphere? To avoid these somewhat ill-defined questions, I considered D and d by their initial definition which we used for 2D for regular polygons. Therefore, I will consider d to be the shortest distance from the center to the boundary of the torus and D as the longest such distance. Finding D and d in the 2-torus requires more involved computations. For this, we will need to use a technique commonly used in calculus, which is described in Theorem 1.

Theorem 1. *For any differentiable function $f(x)$, the minima and maxima of $f(x)$ occur at x such that $f'(x) = 0$.*

Note: The following paragraph is meant to explain Theorem 1. If you are already familiar with this concept, you can safely proceed to the subsequent paragraph knowing that no new information will be introduced here.

Theorem 1 tells us that for some function $f(x)$, we can find the points at which it attains a maximum or a minimum by taking its derivative, setting it equal to zero and then solving for x. Once this is done, we can compute the value of f at these points and find its minima and maxima. For example, if we take the function $f(x) = x^2$, we can find its minimum (it has no finite maximum so we will not get a maximum value) by taking its derivative $f'(x) = 2x$ and solving $2x = 0$. So, we know that we get a minimum at $x = 0$, which shouldn't be surprising since we know that this parabola has its minimum value at zero. This works in a similar way for functions that have multiple (local) maxima and minima. For example, the function $f(x) = \sin(x)$ has minima at x such that $f'(x) = \cos(x) = 0$. This means that it will have a maximum at $x = \frac{\pi}{2} + 2\pi a$ and minima at $\frac{3\pi}{2} + 2\pi a$, where a is an integer.

Theorem 1 can be extended to a function of multiple variables, which we will need to use it in this context. But what function do we want to maximize and minimize? A first intuitive guess might be to choose the equation in (11), but this would not be a good choice because what we want to extremize is the distance from the origin. So, we will choose the distance function:

$$f(x, y, z) := \sqrt{x^2 + y^2 + z^2} \tag{13}$$

We can rewrite this in a more useful way, in terms of our parametrization:

$$f(\theta) = \sqrt{R^2 + r^2 + 2Rr\cos\theta} \tag{14}$$

This turns out to be a function of only one variable, which will simplify the process of finding the maxima and minima. Taking the derivative of the function in (14), we get the following to solve:

$$-2Rr\sin\theta = 0 \tag{15}$$

meaning that the maxima and minima will be at $\theta = 0, \pi$ (since we know θ only goes from 0 to 2π). From this, we compute $f(0)$ and $f(\pi)$, to determine which is the maximum and which is the minimum.

$$f(0) = \sqrt{R^2 + r^2 + 2Rr}$$
$$f(\pi) = \sqrt{R^2 + r^2 - 2Rr} \tag{16}$$

So, we get that $d = \sqrt{R^2 + r^2 - 2Rr}$ and $D = \sqrt{R^2 + r^2 + 2Rr}$, so β is:

$$\begin{aligned}\beta &= \frac{\sqrt{R^2 + r^2 - 2Rr}}{\sqrt{R^2 + r^2 + 2Rr}} \\ &= \frac{\sqrt{(R-r)^2}}{\sqrt{(R+r)^2}} \\ &= \frac{R-r}{R+r}\end{aligned} \tag{17}$$

This result, at first glance, doesn't seem very informative, since we don't have much information about R and r. This leads us to a discussion that I have so far ignored: what is a "regular" torus? In fact, there are infinitely many different tori with a different ratio of $\frac{R}{r}$. This shouldn't matter in general, but for the purposes of looking at the evolution of β in terms of the dimension of the torus, it will be useful to specify the relationship between R and r. When I consider the evolution of β in n-tori, I will assume that all the tori in the progression have the same value of $\frac{R}{r}$, which will be some constant that we will call α, so we have $R = \alpha r$. Let's rewrite β with this notation in hand:

$$\beta = \frac{R - \alpha R}{R + \alpha R} = \frac{1-\alpha}{1+\alpha} \tag{18}$$

Since α is simply a constant, β is itself a constant.

Now that we have computed β for the 2-torus, it is time to define higher dimensional tori. In higher dimensions, the torus is defined such that the sum of the square of each pair of coordinates is equal to 1, which means that the boundary is equidistant from the center at each point. Therefore, β will, by definition, always be 1, for the same reason as for the sphere. This result, while it may seem trivial, will have quite profound implications later on.

ii. Torus-Like Symmetric Objects

A slightly more complex result stems from torus-like objects, which we will call pseudo-tori. In higher dimensions, there are multiple ways to construct pseudo-tori, but the 2D case will remain the torus we studied in the last section. For the purpose of this discussion, I have considered only one of these definitions completely which is discussed below. A possible next step to the project would be to show that this relation holds for all definitions of these objects. [12] Appendix 1 contains a more in-depth discussion about the different ways we can construct pseudo-tori objects and a short justification of the choice I made in using the following equation to define them.

Notation: For an n-pseudo-torus, we require $n+1$ euclidean spacial coordinates. We will call these x_i, such that i runs from 1 to $n+1$. x_1 and x_2 correspond to x and y in our previous notation.

Using the above notation, we will define an n-pseudo-torus by the following equation:

$$(\sqrt{x_1^2 + x_2^2} - R)^2 + \sum_{3}^{n+1} x_i^2 = r^2 \tag{19}$$

We will parametrize this in a clever way such that the sum of the squares of all the variables will always be $R^2 + r^2 + 2Rr\cos\theta$, where θ, R and r are defined in the same way as before. Obviously, we will need to introduce a new angle for each new dimension that we introduce. These angles will be measured from the axis in the new "direction" in the same way as ϕ is measured from the z-axis. Four examples of

[12] As a quick sanity check, β was computed for one other definintions for only the 3D case. β was as expected. The computation and result are not included since it was not done formally.

this parametrization as well as a short explanation of how the parametrization scals with dimension can be found in Appendix 2.

Using the same technique as we used for the 2-pseudo-torus, we will be able to compute β for the 3-pseudo-torus. In fact, given that we have cleverly parametrized our tori so that $\sum_1^{n+1} x_i^2 = R^2 + r^2 + 2Rr\cos\theta$, we can easily compute β for all n-pseudo-tori. We know that the distance function, which we are trying to extremize, is:

$$f(\theta) = \sqrt{R^2 + r^2 + 2Rr\cos\theta} \tag{20}$$

But, we already know the answer to this problem from the 2D case. Since extremizing means setting $\sin\theta = 0$, we know that $x_i = 0$ for all $i \geq 3$. Finally, we know that x_1 and x_2 are the same for any dimension. With all this information, we can find that β is

$$\beta = \frac{1-\alpha}{1+\alpha} \tag{21}$$

for the n-pseudo-torus for all n [13], with α defined as before. This is a very interesting and somewhat surprising result. In fact, in all the cases we have computed so far, β depends on the dimension, except for that of the sphere. This could lead us to think that there might be some underlying link between these shapes and we will discuss this is Section VI.

iii. Cones

Let's start with the 2D cone (or 2-cone). Like the n-torus, the n-cone is embedded in n+1 space.

[13] You can check this yourself, with only the calculus used above and a large quantity of algebra. The exercise is not difficult, but it is quite tedious.

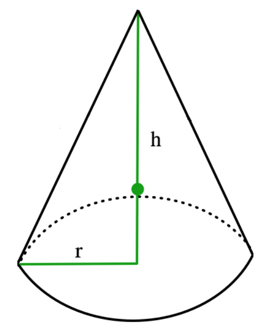

Figure 13: *The Cone.*

Now, we need to parametrize our cone. We define the cone as follows:

$$x^2 + y^2 - z^2 = r^2 \tag{22}$$

and we can parametrize it:

$$\begin{aligned} x &= u\sin\theta\cos\varphi \\ y &= u\sin\theta\sin\varphi \\ z &= u\cos\theta \end{aligned} \tag{23}$$

where the angles φ and θ are shown below and u is the radius of the cone at any given angle.

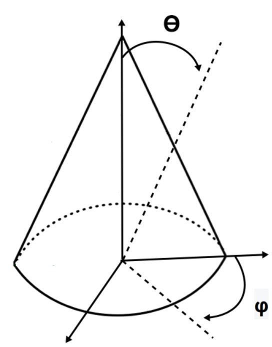

Figure 14: *φ and θ on the cone.*

The center of a cone is not as obvious to find as the center of the previous objects we have studied. To do this, we will find the center of mass, in the usual way. First, let us find the mass, assuming that $r = h$ (we will use this as a definition of a "regular cone" for simplicity, although it shouldn't matter) and constant density with $\rho = 1$:

$$\begin{aligned} M &= \rho \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{h \sec\theta} u^2 du \sin\theta d\theta d\varphi \\ &= 2\pi \int_0^{\frac{\pi}{4}} \frac{h^3}{\sec^3\theta} d\theta \\ &= \frac{\pi h^3}{3} \end{aligned} \tag{24}$$

This is as expected since it is, in fact, the volume of the cone. We know that the cone is radially symmetric, so only the z coordinate of the center of mass will be non-zero. Let's calculate it. We know that:

$$z_{cm} = \frac{1}{M} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{h \sec\theta} z u^2 du \sin\theta d\theta d\varphi \tag{25}$$

With this in hand, let's compute the center of mass coordinate:

$$\begin{aligned} z_{cm} &= \frac{1}{M} \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{h \sec\theta} u^3 du \cos\theta \sin\theta d\theta d\varphi \\ &= \frac{3h}{4} \end{aligned} \tag{26}$$

Note: We center the tip of the cone at the origin and the cone goes up from there, seemingly upside down from the image, which is why the center of mass seems so high. However, the images have been left in the opposite direction for ease of visualization.

Now, we can find β:

$$\beta = \frac{\frac{3h}{4}}{\frac{h}{4}} = \frac{1}{3} \tag{27}$$

since the smallest distance will go from the center of mass to the base of the cone and the longest distance will go from the center of mass to the tip of the cone.

Now, we will look at the n-dimensional case. We define the n-cone as follows:

$$\sum_1^n x_i^2 - x_{n+1} = r^2 \tag{28}$$

The volume will scale according to the following relation, which you can easily prove for yourself using induction and the volume of the n-sphere, which is defined as:

$$V = \frac{\pi^{\frac{n-1}{2}}}{n(\Gamma(\frac{(n+1)}{2} + 1))} h^n \tag{29}$$

where $\Gamma(\frac{(n+1)}{2} + 1)$ is the half-intger gamma function.

From this, we can easily show that the center of mass will be at

$$x_{n+1} = \frac{n}{n+1} h^{n+2} \tag{30}$$

Now, we can get β for an n-cone:

$$\beta = \frac{1}{n+1} \tag{31}$$

iv. Brief Discussion of the Results from Section V

The results in the previous two sections are very interesting indeed. In fact, we have found mostly results consistent with the "one over some power of the dimension" evolution of β, but we have also found our first exceptions to this general rule: the sphere, the torus and the pseudo-torus. Since we have found only these exceptions and that these two exceptions have the exact same behaviour, some red lights should be going off in our minds. Could there be some fundamental reason why this happens? We should expect this to be the case. We will discuss some hypotheses related to this in the following section.

VI. Surfaces of Revolution

When examining the results that we have obtained, one question immediately comes to mind. All but three of the closed form values of β depend on the number of dimensions: those of the sphere, the torus and the pseudo-torus. For the sphere and torus, this result is to be expected since they are defined such that β will always be 1. However, in thinking about β for the pseudo-torus, we can glean some interesting insight about these seemingly multiple different exceptions which all behave in the same way. In the case of the 2D torus and sphere, it is quite obvious that the torus is a surface of revolution of the sphere [14]. In higher dimensions, this is not so obvious. However, we can see that the symmetries are preserved in the same way as in the 2D case. This is true both for the torus and the pseudo-torus since they are the same in 2D.

This realization may lead us to think that there may be something deeper going on. While I have no formal proof of this to offer, it makes intuitive sense. In fact, it is quite clear that rotation preserves distances. If you are not yet convinced, it is interesting to consider the examples of the tetrahedron and the cone. The cone is the surface of revolution of the triangle, a fact of which it should be quite easy to convince yourself. This too is not quite as obvious in higher dimensions, but it is a known fact in geometry so we will take it at face value. This in hand, we see that while not equal, the expressions for β for these two shapes behave in the same way, both roughly inversely proportional to the number of dimensions.

VII. Results

We set out to find a better definition of the concept of a dimension that might lead to a more intuitive understanding of objects with more than three spacial dimensions and we might want to ask if we have succeeded. We have defined a quantity β and computed it for multiple shapes for an arbitrary number of dimensions N. The results of these are shown in Table 1.

Table 1: *Values of β for Various Shapes in N Dimensions*

Shape	2D	3D	4D	ND
Tetrahedron	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{N}$
Cube	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{4}}$	$\frac{1}{\sqrt{N}}$
Octahedron	N/A	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{4}}$	$\frac{1}{\sqrt{N}}$
Sphere	1	1	1	1
Torus	$\frac{R-r}{R+r}$	1	1	1
Pseudo-Torus	$\frac{R-r}{R+r}$	$\frac{R-r}{R+r}$	$\frac{R-r}{R+r}$	$\frac{R-r}{R+r}$
Cone	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{N+1}$

Table 2: *β for Various Regular and Symmetric Shapes*

This quantity β is so far unique, up to mappings that preserve length (for regular polytopes and the radially symmetric shapes we have explored). Therefore, we can now say that a dimension is a *quantity that determines the scaling of β in increasing dimensions for a given shape.*

VIII. Discussion

This quantity idea of a dimension scaling β is not a sufficient nor a complete definition, but it is a starting point and since we were looking for intuition above full mathematical rigor, I believe that this is sufficient to gain a certain understanding of the problem. For this, it is

[14] If you are not familiar with the concept of surfaces of revolution, you can imagine spinning an object around its central axis. In the case of mapping the sphere to the torus, you can imagine putting it onto a stick and rotating it. The surface cut out by the shape is its surface of revolution.

enlightening to consider the following:

Imagine a situation in which we are given the general shape of an object, say a cube, but not its number of dimensions. Now, imagine we are able to measure the distances d and D of this cube [15]. Having simply these pieces of information in hand, we can determine the number of dimensions of the cube.

Can this game be played in reverse? Could we, from β and the number of dimensions, determine the shape we are dealing with? The answer is yes, but we can only identify a family of shapes. Indeed, we have seen that β is not unique, but rather it defines a series of shapes that are related by some map that preserves distances or symmetries.

It is now time to address some more formal questions about these results. I have not presented any formal proofs in this paper and have often relied on intuition and some generalizations which some could call hasty. I have also failed to provide any proofs of uniqueness, which in the case of a quantity we would like to use to qualify the evolution of a shape in increasing dimensions, would be an important piece of information to have. However, I would make the argument that none of these things are truly necessary for the current investigation. In fact, we have a formal proof of the value of β for the cube and we only need one example if we only wish to build some intuition. The computation for other shapes can help strengthen this intuition, but the core information we are given by β doesn't require it. Of course, a more formal approach could uncover further information, but we will save this for another day.

[15] This could be done in a number of ways. The most natural would be to take two particles sent from the center of the cube to its face and corner. If these particles are going at equal speeds, we can get d and D directly since they are simply the travel time of the particles, which is a quantity we have been historically good at measuring. Of course, the logistics of this whole thought experiment are hypothetical and optimistic at best, but they are nonetheless interesting and useful to think about.

IX. Conclusion

By defining and computing β, we have indeed succeeded at finding a new, and as has been argued more intuitive, definition of a dimension. In fact, rather than relying on the orthogonality of directions, we need only consider the scaling of the ratio of two lengths, something which is done frequently in our day to day lives. We also found this β for all regular polygons, which while not directly useful is an interesting artifact of the exploration that was done. In effect, the results we have obtained is a type of game, where we can find the geometry of an object, its number of dimensions and its β if we are given the other two quantities. Moving forward, a more in-depth exploration of the hypothesis regarding the conservation of β for shapes which share symmetries would strengthen the results. In addition, a more formal presentation of the results obtained would add to their legitimacy, when regarded from a purely mathematical perspective.

Appendix 1: Toratopic Notation and Spheration

Toratopic notation is used as a short-hand to encode the information about how toratopes (torus-like objects) are constructed in higher dimensions [Tororopic Notation]. It relies on a series of vertical lines, some of which are placed in parentheses, the sum of which represents the number of dimensions of the object. The lines inside and outside the parentheses represent two different parts of the construction of a toratope, which we define below.

Definition 7. *The lines outside the parentheses represent **digons**, line segments. They are, naturally, one-dimensional objects.*

Definition 8. *The lines inside the parentheses represent **spherations**. A spheration can be understood as a way to lift digons from one to multiple dimensions, by essentially tracing along the length of the digon with the surface of an n-sphere.*

To better understand these definitions, let us consider the 2D torus, which is represented (II)I in toratopic notation. Recall the implicit definition of the 2-torus:

$$(\sqrt{x^2+y^2}-R)^2+z^2=r^2 \tag{32}$$

where R and r are the large and small radii of the torus respectively. From this definition, it is immediately evident that we have the expression of a circle within it. This is the spheration component (II). We can also see this since it contains two of the variables. The z component represents the digon. So, in terms of toratopic notation, we are spherating the digon by tracing the length of the z component (which will form a loop) with a circle, formed by the x and y components.

Let us continue with another example, which is the form of the pseudo-torus which was considered in section V(ii). It is denoted (II)II and its implicit definition is:

$$(\sqrt{x^2+y^2}-R^2)+z^2+w^2=r^2 \tag{33}$$

with R and r defined as before. Here, we have two dimensions of digons and two dimensions of spheration, which manifests as one two-dimensional spheration about two digons. We can see this from the notation, as well as the equation, where the x and y term is again the spheration and the z and w components are the two digons. We construct the 3-pseudo-torus by running the spheration (which is a circle in this case) along both the digons. Both digons will again be loops. As may be expected, as dimensions are added, we will add digons and conserve the two dimensions of spheration.

Other ways to construct toratopes in higher dimensions exist, and in fact the number grows with each new dimension added proportionally to the number of possible permutations of the number of digons and spherations required. The 3-pseudo-torus can be constructed in four ways: (II)II, (III)I, (II)(II) and ((II)I)I. It was mentioned that one other case was investigated for the 3D case. For this, the (III)I construction was used. We define this as:

$$(\sqrt{x^2+y^2+z^2}-R)^2+w^2=r^2 \tag{34}$$

again, with R and r as before.

We should now discuss why the (II)II construction was used. First, it is important to note that only the (II)II and (III)I constructions retain the general R and r we have been using, which limited our initial decision to one of these. The choice between these two was made in order to simplify the computations as well as retain the symmetries of the 2-torus. It is also the most natural choice for its evolution in higher dimensions. By choosing this construction, we can always choose the construction with only two dimensions of spheration and ensure that the evolution in higher dimensions is smoothe.

Appendix 2: Parametrization of the n-Psuedo-Torus

In order to get a useful parametrization of the pseudo-torus we will begin with the 2-pseudo-torus and multiply the first 2 variables by the sine of some angle. The 3^{rd} variable will be r times the cosine of the angle. Then we can apply this logic recursively for higher dimensional pseudo-tori. Four examples are provided to illustrate this evolution of parametrization.

The 3-pseudo-torus is parametrized as follows, with the angle ζ measured from the x_4 axis:

$$\begin{aligned} x_1 &= (R+r\cos\theta)\cos\varphi \\ x_2 &= (R+r\cos\theta)\sin\varphi \\ x_3 &= r\sin\theta\sin\zeta \\ x_4 &= r\sin\theta\cos\zeta \end{aligned} \tag{35}$$

The 4-pseudo-torus is parametrized as follows, with the angle η measured from the x_5

axis:

$$\begin{aligned} x_1 &= (R + r\cos\theta)\cos\varphi \\ x_2 &= (R + r\cos\theta)\sin\varphi \\ x_3 &= r\sin\theta\sin\zeta\sin\eta \\ x_4 &= r\sin\theta\cos\zeta\sin\eta \\ x_5 &= r\sin\theta\cos\eta \end{aligned} \tag{36}$$

The 5-pseudo-torus is parametrized as follows, with the angle ω measured from the x_6 axis:

$$\begin{aligned} x_1 &= (R + r\cos\theta)\cos\varphi \\ x_2 &= (R + r\cos\theta)\sin\varphi \\ x_3 &= r\sin\theta\sin\zeta\sin\eta \\ x_4 &= r\sin\theta\cos\zeta\sin\eta \\ x_5 &= r\sin\theta\cos\eta\sin\omega \\ x_6 &= r\sin\theta\cos\eta\cos\omega \end{aligned} \tag{37}$$

The 6-pseudo-torus is parametrized as follows, with the angle ψ measured from the x_7 axis:

$$\begin{aligned} x_1 &= (R + r\cos\theta)\cos\varphi \\ x_2 &= (R + r\cos\theta)\sin\varphi \\ x_3 &= r\sin\theta\sin\zeta\sin\eta\sin\psi \\ x_4 &= r\sin\theta\cos\zeta\sin\eta\sin\psi \\ x_5 &= r\sin\theta\cos\eta\sin\omega\sin\psi \\ x_6 &= r\sin\theta\cos\eta\cos\omega\sin\psi \\ x_7 &= r\sin\theta\cos\eta\cos\psi \end{aligned} \tag{38}$$

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Note to the reader: the sources below were used only for preliminary research. I include them because they are useful as introductory references for anyone who is curious about the topic, however any information taked from these sources was verified throuh one or more of the sources above.

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