Continuum Theory of an Ionized Inductively Coupled Plasma

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1 Introduction

Inductively coupling is a method used to create a controlled plasma which is used for various applications in research.

The origin of inductively coupled plasma (ICP) is linked to another form of plasma generation called arc-jet plasmas. In the early days of space exploration, the U.S. began using arc-jet facilities to characterize materials used for space re-entry. These are called thermal protection systems (TPS). The facilities housing these arc-jets were expensive to run, large, and had various other drawbacks.

During this time period, the USSR was actively competing with the U.S. for space supremacy. The soviets needed a way to test and characterize materials for space re-entry just as the U.S. was doing, but the cost of such facilities was a hindrance. ICP facilities were a direct result of this. They, generally, are much less expensive to run, are fairly simple, and can have a much smaller footprint. For these reasons, it has been possible for smaller groups with less funding to build ICP facilities. Testing is also able to be carried out in both types of facilities, allowing for more comprehensive data.

2 What is an Inductively Coupled Plasma

Inductive coupling is one method in which a plasma can be created from a gas. The general method by which a plasma is created is the same for all types of plasma. It occurs when electrons are stripped from a molecule (ionization) and when molecules are broken into their constituent atomic components (dissociation).

The method by which this is achieved is through a time varying magnetic field. Gas passes through this magnetic field which is created from an outside power source. Energy is subsequently transferred from the magnetic field to an oscillating electric field within the plasma. Energy is transferred from the electric field to the gas, accelerating and thermalizing the particles.[8] This process is illustrated in Figure 1.

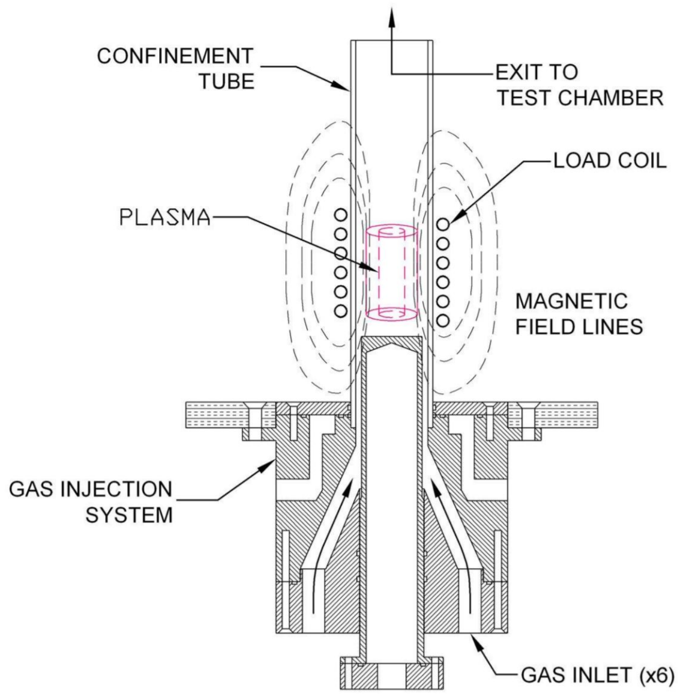

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Figure 1: Plasma generating components of an ICP facility.

There are two processes by which the above occurs. The first is Joule Heating. This is the process by which an electric current passing through a conductor dissipates energy losses in the form of heat. This is shown in Equation 1,

$$Q = I^2RT \tag{1}$$

where Q is the amount of heat released, I is the current, R is the resistance, and T is time.[8]

The second process by which energy is transferred is called hysteresis. The electrons present in the gas spin and create a localized magnetic field, called a magnetic dipole. The force created from this dipole can be described by the Lorentz Law, seen in Equation 2.

$$\vec{F} = q(\vec{E} + (\frac{\vec{v}}{c}) \times \vec{B}) \tag{2}$$

where F is the force on the electron, q is the charge, v is velocity, E is the electric field, and B is the magnetic field.[8]

As discussed earlier, a large magnetic field is created with a load coil loop. This magnetic field oscillates and rapidly flips orientation. This flipping of the magnetic field induces a large torque on the electron dipole. The high frequency of oscillations causes the amplitude of the field to grow and eventually the atom ionizes.[8]

3 Plasma Continuum Based on Model Equations

The following method of determining the characteristics of an inductively coupled plasma will employ a multitude of equations which each describe different key components of plasma generation. This method is advantageous because it uses different relations to describe the plasma. Each of these relations are useful in their own respective right.

The equations that will be employed include mass conservation equations for electrons and ions, as well as an electron-energy equation. There will also be an electron-energy equation which typically describes the applied power. Maxwell's equations are used to compute the electric field which is then used to compute the power of the plasma. To determine the plasma edge conditions, a Bohm criterion is applied. The Bohm criterion is used to describe power coupling as well as plasma generation. The Navier-Stokes equation is also used along with a gas energy equation so as to preserve self-consistency.[1]

3.1 Conservation of Mass

The convservation of mass in an inductively coupled plasma is described by Equation 3,

$$\frac{\partial \rho_s}{\partial t} + \bigtriangledown \cdot \rho_s \mathbf{u} = - \bigtriangledown \cdot \mathbf{J}_s + \sum_{r=1}^{nr} R_{sr} \tag{3}$$

where ρ_s is the species density, $\mathbf{u}$ is the mass averaged velocity of the mixture, and $\mathbf{J}_s$ is the diffusion mass flux due to gradients in the species density, pressure, and electrostatic potential. Also, R_{sr} represents the mass rate of production of the species s from the reactor r, where the "reactor" includes all the plasma generating components.[1]

The chemical reaction that takes place, which is described within Equation 3, must also be shown. This is the rate of species production and is seen in Equations 4 and 5,

$$\sum_{s=1}^{ns} v'_{sr}\chi_s \rightarrow \sum_{s=1}^{ns} v''_{sr}\chi_s, \quad r = 1, nr \tag{4}$$

$$R_{sr} = M_s(v''_{sr} - v'_{sr})k_r \prod_{k=1}^{ns}(\frac{\rho_k}{M_k})^{v'_{kr}} \tag{5}$$

where χ_s and M_s are the chemical symbol and molecular weight of the species s, respectively. v''_{sr} and v'_{sr} are stoichiometric coefficients in Equation 4.[1]

3.2 Conservation of Momentum

The next equation that is used is the conservation of momentum. Typically, the conservation of momentum is solved using the Navier-Stokes equation. This can be seen in Equation 6.[7]

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \nabla \cdot (\mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) - \frac{2}{3}\mu(\nabla \cdot \mathbf{u})\mathbf{I}) + \mathbf{F} \tag{6}$$

Assuming that the only external force applied is that of gravity and given $\prod$ is a viscous stress tensor, the Navier-Stokes equation can be simplified slightly to Equation 7,

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot \rho \mathbf{u}\mathbf{u} = -\nabla p + \nabla \cdot \prod + \rho g \tag{7}$$

where p is the total pressure. The total pressure is found by summing the partial pressures of the neutrals, ions, and electrons. This can be seen in Equation 8.[1]

$$p = \sum_{s \neq e,i} \rho_s \frac{R_g}{M_s} T + \sum_{s=i} \rho_s \frac{R_g}{M_s} T_i + \rho_e \frac{R_g}{M_e} T_e \tag{8}$$

Normally, electron pressure may not be taken into consideration when calculating total pressure in other systems. However, the high level of ionization in a plasma causes the ratio of electron partial pressure to the total pressure to be significantly higher.[1]

3.3 Energy Transport

A plasma generally reaches very high temperatures. In an ICP generation scenario, certain parts of the plasma, namely the coupling region, can reach temperatures as high as the surface of the sun. With such high temperatures, it is important to take into account thermal transport of energy. Equations 9 and 10 describe this thermal transport.[1]

$$\begin{aligned} \tfrac{\partial \rho_{\mathrm{n}} e_{\mathrm{n}}}{\partial t} + \nabla \cdot (\rho_{\mathrm{n}} e_{\mathrm{n}} \mathbf{u}) = -\nabla \cdot \sum_{\text{neutrals}} (\mathbf{q}_{\mathrm{s}} + h_{\mathrm{s}} \mathbf{J}_{\mathrm{s}}) - p_{\mathrm{n}} \nabla \cdot \mathbf{u} \\ + \sum_{\text{neutrals}} \tfrac{\mathbf{J}_{\mathrm{S}}}{\rho_{\mathrm{S}}} \cdot \nabla p_{\mathrm{s}} + \sum_{\text{neutrals}} Q_{\mathrm{es}} + \sum_{\text{neutrals}} Q_{\mathrm{ce,s}} \\ + \sum_{r} \sum_{\text{neutrals}} R_{\mathrm{sr}} H_{\mathrm{sr}} \end{aligned} \tag{9}$$

$$\frac{\partial \rho_e e_e}{\partial t} + \nabla \cdot (\rho_e e_e \mathbf{u}) = -\nabla \cdot (\mathbf{q}_e + h_e \mathbf{J}_e) - p_e \nabla \cdot \mathbf{u} + \frac{\mathbf{J}_e}{\rho_e} \cdot \nabla p_e - \sum_{s \neq e} Q_{es} + P_{ext} + \sum_r R_{er} H_{er} \tag{10}$$

In these equations the thermal energy and the pressure of the neutrals is defined as $\rho_n e_n = \sum_{s \neq e,i} \rho_s c_{v,s} T$ and $p_n = \sum_{s \neq e,i} \rho_s (R_g/M_s) T$. Also, the thermal energy of electrons in terms of its specific heat is defined as $\rho_e e_e = \rho_e c_{v,e} T_e$. The specific enthalpy is $h_s = C_{p,s} T_s$ and the flux which is caused by thermal conduction is defined as $\mathbf{q}_s = -k_s \nabla T_s$.[1]

The energy exchange due to elastic collisions between the species s and electrons is described by Equation 11.[1]

$$Q_{\mathrm{es}} = \frac{3}{2} \frac{R_{\mathrm{g}}}{M_{\mathrm{e}}} \rho_{\mathrm{e}} \left(T_{\mathrm{e}} - T\right) f_{\mathrm{s}} \frac{2 M_{\mathrm{e}}}{M_{\mathrm{s}}} s \neq e \tag{11}$$

The other term that will be defined is the energy transfer due to charge-exchange collisions between the neutrals and the ions. This can be seen in Equation 12.[1]

$$Q_{\mathrm{ce,s}} = \sum_{i=\mathrm{ions}} \frac{3}{2} \rho_{\mathrm{i}} \frac{R_{\mathrm{g}}}{M_{\mathrm{i}}} \left(T_{\mathrm{i}} - T\right) f'_{\mathrm{is}} \tag{12}$$

Here, f_s describes the frequency of the elastic-collisions between the neutral species s and the electrons. Meanwhile, f'_s is the frequency of the charge-exchange between ions i and the neutral species s. The power deposition inside of the plasma which is caused by external forces is described by P_{ext}. Finally, H_{sr} is the increase in the thermal energy of the species s which is caused by the reaction r per unit mass production of the species s.[1]

3.4 Maxwell's Equations

Typically, ICP chambers are built in such a way so that the plasma is generated as a torch in the vertical direction. They are also housed within a chamber that is cylindrical where the plasma generation occurs in the center of the cylinder. Maxwell's equations are used to determine the electric fields within the chamber.

By solving a simplified form of Maxwell's equations for axisymmetric fields, the azimuthal plasma current can be determined. This is seen in Equation 13,

$$\nabla^2 E_\theta = i\omega\mu_0\sigma E_\theta \tag{13}$$

where the driven frequency is ω and the permeability constant is μ_0. Assuming fully collisional heating, the azimuthal plasma current can be determined as well as the power deposition density. This can be seen in Equation 14,

$$\begin{gathered} j_\theta = \sigma E_\theta \\ P_{ext} = \tfrac{1}{2}\sigma_r |E_\theta|^2 \end{gathered} \tag{14}$$

where σ_r is the real component of the conductivity of the plasma which is σ.[1]

3.5 Diffusion with Stefan-Maxwell Relations

The generation of the plasma in an ICP comes from the injection of various gas mixtures. This typically starts out as pure Argon and then other gases can be added in desired amounts. The diffusion of these gases is important as it is beneficial to know how the gases are moving relative to each other. For instance, one type of gas could be more dominant in the center of the plasma stream while another may be more dominant towards the edges. If this was occurring during an experiment, it would be crucial to know.

To model diffusion, solving a linear system of Stefan-Maxwell relations that govern the diffusion velocities is necessary. The fluxes in the diffusion must equal zero for the model to work. A diffusion formulation with self-consistent effective binary diffusion for a multicomponent plasma where the fluxes are equal to zero can be seen in Equation 15,

$$\begin{aligned} \mathbf{J}_s = &-\frac{pM_s\mathscr{D}_s}{R_\mathrm{g}T}\mathbf{H}_\mathrm{s} + y_\mathrm{s}\frac{p}{R_\mathrm{g}T}\sum_{j\neq e}\mathscr{D}_j M_\mathrm{j}\mathbf{H}_\mathbf{j} \\ &+\left(\frac{q_\mathrm{s}\mathrm{p}_\mathrm{s}M_\mathrm{s}\mathscr{D}_\mathrm{s}}{R_\mathrm{g}T} - y_\mathrm{s}\frac{1}{R_\mathrm{g}T}\sum_{j\neq e}q_j\rho_j M_j\mathscr{D}_j\right)\mathbf{E} \quad (s\neq e) \end{aligned} \tag{15}$$

where $\mathbf{H}_s$ is the main driving force because of nonelectromagnetic effects. These effects include density, temperature, and pressure gradients, as well as $\mathbf{E}$ which is the electric field.[3,4,5] These terms can be seen in Equations 16 and 17. The diffusion coefficient can be seen in Equation 18.[1]

$$\mathbf{H}_s = \nabla(\frac{p_s}{p}) \tag{16}$$

$$\mathbf{E} = \frac{\nabla p_e}{q_e \rho_e} \tag{17}$$

$$\mathscr{D}_s = \frac{(1 - x_s)}{\sum\limits_{k=s,e} x_k / \mathscr{D}_{sk}} (s \neq e) \tag{18}$$

In Equation 18 and 15, x_s and y_s are the molar and mass fractions, respectively. This diffusion is also for a species s in a mixture.

Next, the diffusion of electrons must be accounted for. This is separate from the diffusion of a species in a mixture, and is done in the axial and radial directions, as an ICP is created in a cylindrical shape. This can be seen in Equation 19,

$$\mathbf{J}_e = \frac{1}{q_e} \sum_{s \neq e} q_s \mathbf{J}_s \tag{19}$$

where the charge per unit mass is q_s of the species s.[1]

3.6 Boundary Conditions

Just as in any flow, the boundary conditions at the edge of the flow in an ICP must be specified for a model to be valid. The conditions at the boundary are much different than those within the center of the plasma stream. Assuming the flow is subsonic, all conditions of the flow must be specified except for one. The last remaining condition may be extrapolated. The model outlined above specifies gas and electron temperatures, gas composition, and mass flow rate at the inlet of the plasma. The extrapolated condition is the total pressure.[1]

At the exit, the total pressure is known. Other variables are extrapolated from the interior conditions such as temperature, density, and velocities.

Assuming that the ICP chamber acts as a reactor of sorts, the mass flux is determined from a sticking coefficient. This can be seen in Equation 20, which is for recombining neutral species.[1]

$$\mathbf{J}_s = -\frac{1}{4} v_{th,s} \rho_s S_s \tag{20}$$

A similar equation is determined but for positive ions recombining with electrons. This can be seen in Equation 21,

$$\mathbf{J}_s = v_{B,s} \rho_s S_s \tag{21}$$

where the ion Bohm velocity is described by $v_{B,s}$ which is $\sqrt{(R_g T_e/M_s)}$. The sticking coefficient is S_s and the average thermal velocity $v_{th,s}$ is $(\sqrt{(8R_g T/\pi M_s)})$ for species s.[1]

4 Plasma Continuum Based on Charge Density Function

Another method for the description of a plasma in full equilibrium is with the charge density function, $n(r)$. The charge density function is able to accurately describe long range correlations of the Coulomb interaction.[2]

The probability density function of the net charge density is a random function. This function can be used on the charge fluctuations in the plasma to determine the free energy of the plasma.[2]

4.1 Potential Distribution of a System in Equilibrium

The relationship between the chemical potential of a system in equilibrium and the probability distribution function can be seen in Equation 22,

$$\mu = kT \ln \frac{N}{V} \Lambda^3 - kT \ln \int_{\phi=-\infty}^{\infty} d\varphi P(\varphi) e^{-q\varphi/kT} \tag{22}$$

where μ is the chemical potential and $P(\varphi)$ is the probability distribution function at a point.[6] Using this equation it is possible to derive the Debye-Hückel expression which describes the free energy of a plasma.

Equation 22 is true for any fluid where the interaction energy can be written in the form seen in Equation 23,

$$U = \sum_{i<j} q_i q_j u(\mathbf{r}_i - \mathbf{r}_j) \tag{23}$$

where $u(\mathbf{r}_i - \mathbf{r}_j)$ is the two body interaction and the q_i terms are coupling parameters of the particles.[6]

The probability distribution function seen in Equation 22 must also be defined. This can be seen in Equation 24,

$$\begin{aligned} P\left[\varphi\left(\mathbf{r}_0\right)\right] = \int d\mathbf{r}_1 \cdots d\mathbf{r}_N \delta & \left[\varphi\left(\mathbf{r}_0\right) \right. \\ & \left. - \sum_{i=1}^{N} q_i u\left(\mathbf{r}_0 - \mathbf{r}_i\right)\right] P\left(\mathbf{r}_1, \cdots \mathbf{r}_N\right) \end{aligned} \tag{24}$$

where $\varphi(\mathbf{r}_0) = \sum_{i=1}^{N} q_i u(\mathbf{r}_0 - \mathbf{r}_i)$ is defined as the potential within the fluid at a single point.[6]

The equilibrium probability distribution function of the positions of the particles is seen in Equation 25.[6]

$$P\left(\mathbf{r}_1, \cdots \mathbf{r}_N\right) = \frac{1}{Z_N} e^{-\beta U(r_1, \cdots r_N)} \tag{25}$$

Equation 26 defines Z_N. Combining Equations 24 and 25 yields Equation 27.[6]

$$Z_{\mathrm{N}} = \int d\mathbf{r}_1 \cdots d\mathbf{r}_{\mathrm{N}} e^{-gU(r_1,\cdots r_N)} \tag{26}$$

$$\begin{aligned} P\left[\varphi\left(\mathrm{r}_0\right)\right] = \frac{1}{Z_N} \int d\mathbf{r}_1 \cdots d\mathbf{r}_N \delta\left(\varphi\left(\mathbf{r}_0\right) - \sum_{j=1}^{N} q_i u\left(\mathbf{r}_0 - \mathbf{r}_i\right)\right) \\ \cdot \exp\left[-\beta \sum_{i<j} q_i q_i u\left(\mathbf{r}_i - \mathbf{r}_i\right)\right] \end{aligned} \tag{27}$$

If an integration of $P(\varphi)$ is done as in Equation 28, then Equation 27 becomes Equation 29.[6]

$$\int_{\varphi=-\infty}^{+\infty} e^{-e_0\beta\varphi} P(\varphi) d\varphi \tag{28}$$

$$\begin{aligned} \int_{\varphi=-\infty}^{+\infty} e^{-q_0\varphi} P(\varphi) d\varphi = \frac{1}{Z_N} \int d\mathbf{r}_1 \cdots d\mathbf{r}_N \\ \cdot \exp\left[-\beta \sum_{j=1}^{N} q_0 q_i u\left(\mathbf{r}_0 - \mathbf{r}_i\right) - \beta \sum_{i<j=1}^{N} q_i q_j u\left(\mathbf{r}_i - \mathbf{r}_j\right)\right] \end{aligned} \tag{29}$$

Carrying out the integral, seen in Equation 30, taking into account that the free energy is related to the configuration integral, as shown in Equation 31, and adding the definition of the chemical potential energy which is seen in Equation 32, Equation 22 is proven to be true.[6]

$$\int_{\varphi=-\infty}^{+\infty} e^{-\beta q_0 \varphi} P(\varphi) d\varphi = \frac{1}{V} \frac{Z_{N+1}}{Z_N} \tag{30}$$

$$F_N = -kT \ln \left(\frac{Z_N}{N! \Lambda^3} \right) \tag{31}$$

$$\mu = \lim_{N \to \infty} (F_{N+1} - F_N) = -kT \ln \left(\frac{Z_{N+1}}{N \Lambda^3 Z_N} \right) \tag{32}$$

4.2 Description of Plasma Continuum

In this approach to modeling a plasma continuum, the average interactions between all particles are used with the net charge density function ($n(\mathbf{r})$) described earlier. ($n(\mathbf{r})$) is considered to be a continuous distribution of charge throughout the plasma. This begins with taking a finite small region within the plasma (n_j), finding the net charge density within this volume, and applying this small region to the entire continuum limit.[2]

The finite volume that is chosen must be large enough to contain enough particles that a continuum can be assumed. It is also necessary for there to be an equal number of positive and negative particles. However, this assumption cannot be made. For this to be true, an approximation is made by the product of combinatorial and volume factors with the average Maxwell-Boltzmann factor. The Maxwell-Boltzmann factor is $\exp\left(-\overline{E}/kt\right)$. $\overline{E}$ is the average

Coulomb energy for the given occupation numbers. This yields Equation 33 and 34, where N_+ and N_- are the number of positive and negative particles.[2]

$$P\left(N_{1+}, N_{1-}; \cdots N_{j+}, N_{j-}; \cdots\right) \cong \frac{1}{Z}\frac{N_+!}{\prod_j (N_{j+}!)}\frac{N_-!}{\prod_i \left(N_{j-}!\right)} \prod_j v_j^{N_{j+}+N_{j-}} e^{-\beta U(N_{j+}N_{j-}\cdots N_{j+},N_{j-}\cdots)} \tag{33}$$

$$e^{-\beta U(N_{j+},N_{j-};\cdots N_{j+}N_{j-};\cdots)} = \prod_{j\neq k} \exp\left[-\frac{\beta e^2}{2}\frac{(z_+N_{j+}-z_-N_{j-})(z_+N_{k+}-z_-N_{k-})}{\langle r_{ik}\rangle}\right] \cdot \prod_{j=k}\left[-\frac{\beta e^2}{2}\left[\frac{\left(z_+N_{j+}-z_{-N_i-1}\right)^2}{\langle r_{jj}\rangle} - \frac{z_+^2N_{j+}^2+z_-^2N_{j-}}{\langle r_{jj}\rangle}\right]\right] \tag{34}$$

Additionally, Equation 35 shows

$$\frac{1}{\langle r_{ik}\rangle} = \frac{1}{v_i v_k}\int_{r_j\subset v_j} dr_j \int_{r_k\subset v_k} dr_k \frac{1}{|r_j - r_k|} \tag{35}$$

which can be used in the previous equation.[2]

The number of positive and negative particles are now determined. The average number of the positive and negative particles is $\overline{N}_{j+}$ and $\overline{N}_{j-}$, respectively. This value is not a constant in every finite volume within the continuum however, so the error associated with the positive and negative fluctuations of these constants can be described with Equation 36.[2]

$$\Delta_{j+} = N_{j\pm} - \bar{N}_{j\pm} \tag{36}$$

The Sterling formula can be used to expand the combinatorial factorials about their maximum values, shown in Equation 37,

$$\begin{aligned} &P\left(\Delta_{1+}, \Delta_{1-}; \cdots \Delta_{j+}\Delta_{j-}; \cdots\right) \\ &= \tfrac{N_+! N_-!}{Z} \left\{ \prod_j \tfrac{(v_j)^{\overline{N}_{j+} + \overline{N}_{j-}}}{(N_{j+}!)(N_{j-}!)} \exp\left[-\tfrac{\Delta_{j+}^2}{2\bar{N}_{j+}} - \tfrac{\Delta_{j-}^2}{2\bar{N}_{j-}} \right] \right\} \\ &\cdot \exp\left[-\tfrac{\beta e^2}{2} \sum_{j,k} \tfrac{(z_+\Delta_{j+} - z_-\Delta_{j-})(z_+\Delta_{k+} - z_-\Delta_{k-})}{\langle r_{jk} \rangle} \right] \\ &\cdot \exp\left[\tfrac{\beta e^2}{2} \sum_j \tfrac{\left(z_+^2 N_{j+} + z_-^2 N_{j-}\right)}{\langle r_{jj} \rangle} \right] \end{aligned} \tag{37}$$

To make things easier the previous definitions are transformed to new variables. This can be seen in Equations 38, 39, and 40. The result of the integration can be seen in Equation 41. Equation 41 can be rewritten in the continuum limit and is shown in Equation 42.[2]

$$n_i = \left(z_+\Delta_{j+} + z_-\Delta_{j-}\right) / v_j \tag{38}$$

$$s_j = \left[\left(z_+ z_-\right)^{\frac{1}{2}} / \mathfrak{N} v_j \right] \left(\Delta_{j+} + \Delta_{j-}\right) \tag{39}$$

$$\mathfrak{N} \equiv z_+^2 \boldsymbol{N}_+/\boldsymbol{V} + \boldsymbol{z}_-^2 \boldsymbol{N}_-/V \tag{40}$$

$$P\left(n_1 \cdots n_j \cdots\right) = \frac{\exp\left\{-\sum_j \frac{n_j^2}{2\mathfrak{N}} v_j - \frac{\beta e^2}{2} \sum_{j,k} \frac{n_j n_k}{\langle r_{jk}\rangle}\ v_j v_k\right\}}{\int \cdots \int dn_1 \cdots dn_j \cdots \exp\left\{-\sum_j \frac{n_j^2}{2\mathfrak{N}} v_j - \frac{\beta e^2}{2} \sum_{j,k} \frac{n_j n_k}{\left\langle r_{jk}\right\rangle} v_j v_k\right\}} \tag{41}$$

$$P[n(\mathbf{r})] = \frac{\exp\left\{-\frac{1}{2\mathfrak{N}} \int n(\mathbf{r})^2 d\mathbf{r} - \frac{\beta e^2}{2} \iint \frac{n(\mathbf{r})n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r} d\mathbf{r}'\right\}}{\int dn(\mathbf{r}) \exp\left\{-\frac{1}{2\mathfrak{N}} \int n(\mathbf{r})^2 d\mathbf{r} - \frac{\beta e^2}{2} \iint \frac{n(\mathbf{r})n(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r} d\mathbf{r}'\right\}} \tag{42}$$

With the continuum limit approach, the Coulomb potential at a point in the plasma can be found, as seen in Equation 43, while the distribution function of the potential can be found from the distribution of $n(\mathbf{r})$ according to Equation 44.[2]

$$\varphi = e \int \frac{n(\mathbf{r})}{|\mathbf{r} - \mathbf{R}|} d\mathbf{r} \tag{43}$$

$$P(\varphi) = \int \delta\left(\varphi - e \int \frac{n(\mathbf{r}) d\mathbf{r}}{|\mathbf{r} - \mathbf{R}|}\right) P[n(\mathbf{r})] dn(\mathbf{r}) \tag{44}$$

Working through this equation with the integral representation of the delta function and using Equation 42, Equation 45 is determined.[2]

$$P(\varphi) = \frac{\int_{-\infty}^{\infty} dn(\mathrm{r}) \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \exp\left\{ i\alpha \left(\varphi - e \int \frac{n(\mathrm{r})d\mathrm{r}}{r} \right) - \frac{1}{2\mathfrak{N}} \int n(\mathrm{r})^2 d\mathrm{r} - \frac{\beta e^2}{2} \iint \frac{n(\mathrm{r})(\mathrm{r}')}{|\mathrm{r}-\mathrm{r}'|} d\mathrm{r} d\mathrm{r}' \right\}}{\int dn(\mathrm{r}) \exp\left\{ -\frac{1}{2\mathfrak{N}} \int n(\mathrm{r})^2 d\mathrm{r} - \frac{\beta e^2}{2} \iint \frac{n(\mathrm{r})n(\mathrm{r}')}{|\mathrm{r}-\mathrm{r}'|} d\mathrm{r} d\mathrm{r}' \right\}} \tag{45}$$

The right hand side quotient integrals may be calculated using a method based on a Gaussian integrals formula, shown in Equation 46, as well as the fact that the right hand side exponent is the stationary value of the left hand side exponent. There is no linear term therefore the exponent in the denominator is zero. The variational equation, which makes the exponent in the numerator stationary is seen in Equation 47.[2]

$$\begin{aligned} &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_N e^{-\mathbf{b}\cdot\mathbf{x}} e^{-\mathbf{x}\cdot\mathbf{A}\cdot\mathbf{x}} \\ &\qquad = \frac{\pi^{N/2}}{(|\mathbf{A}||)^{\frac{1}{2}}} \exp\left\{ \frac{1}{\mathbf{4}} \mathbf{b} \cdot \mathbf{A}^{-1} \cdot \mathbf{b} \right\} \end{aligned} \tag{46}$$

$$\beta e^2 \int \frac{n^{(1)}\left(\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' + \frac{n^{(1)}(\mathbf{r})}{\mathfrak{N}} = -\frac{ie\alpha}{r} \tag{47}$$

Equation 47 can be denoted as $n^{(1)}(\mathbf{r})$, and its solution can be seen in Equation 48,

$$n^{(1)}(r) = -i\alpha\mathfrak{N}ee^{-\kappa r}/r \tag{48}$$

where $k = (4\pi\beta e^2\mathfrak{N})^{1/2}$ is the inverse Debye length.[2] This equation can be substituted into the numerator of Equation 45 to obtain Equation 49.[2]

$$P(\varphi) = \frac{1}{2\pi}\int d\alpha e^{i\alpha\varphi} e^{-\alpha^2 \kappa kT/2} = \frac{e^{-\varphi^2/2\kappa kT}}{(2\pi\kappa kT)^{\frac{1}{2}}} \tag{49}$$

It was noted earlier that there are an equal number of positive and negative charges. Equation 49 now needs to be broken up into these positive and negative components to obtain the two chemical potentials. These will be known as μ_+ and μ_-. Using the right hand side of Equation 22 and calling it μ', Equation 50 and 51 are found.[6]

$$\begin{aligned} \mu'_+ &= -kT \\ &\cdot \ln\left\{ \int_{-\infty}^{\infty} d\varphi \frac{\exp[-z_+ e\varphi/kT]\exp\left[-\varphi^2/2\kappa kT\right]}{(2\kappa kT)^{\frac{1}{2}}} \right\} \end{aligned} \tag{50}$$

$$\begin{aligned} \mu'_- &= -kT \\ &\cdot \ln\left\{ \int_{-\infty}^{\infty} d\varphi \frac{\exp[z_- e\varphi/kT]\exp\left[-\varphi^2/2\kappa kT\right]}{(2\kappa kT)^{\frac{1}{2}}} \right\} \end{aligned} \tag{51}$$

From these equations it can be found that $\mu'_+ = -\frac{1}{2}z_+^2 e^2\kappa$ and $\mu'_- = -\frac{1}{2}z_-^2 e^2\kappa$. The number of positive and negative particles are equal therefore $\mu_+ = \partial F/\partial N_+$ and $\mu_- = \partial F/\partial N_-$. From the described relation above Equation 52 can be found.[6]

$$\frac{1}{Z_+^2}\frac{\partial F}{\partial N_+} = \frac{1}{Z_-^2}\frac{\partial F}{\partial N_-} \tag{52}$$

Using this equation, the interaction part of the free energy can be found and is shown in

Equation 53. This equation is the Debye-Hückel function which was described briefly earlier.

$$F = -\frac{1}{3}\left(z_+^2 N_+ + z_-^2 N_-\right) e^2 \kappa \tag{53}$$

5 Conclusion

The objective of this paper was to investigate models for describing an inductively coupled plasma. One of these methods consisted of taking a multi-equation approach, while the other stemmed from a single function known as the Charge Density Function.

The first method mentioned above consisted of model equations which included conservation of mass, conservation of momentum, energy transport, Maxwell's Equations, Stefan-Maxwell relations, and boundary conditions. These key components of plasma generation each describe one portion of the plasma, but together create an accurate model.

The Charge Density Function method for describing a plasma as a continuum was a much more complicated approach, but because it is based on one relation it may be easier to use in some scenarios.

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